Lecture 15

Modeling the Inner Magnetosphere
The Inner Magnetosphere

- The inner magnetosphere includes the ring current made up of electrons and ions in the 10-200keV energy range, the radiation belts with MeV energies and the plasmasphere.
  - The ring current carries a very large fraction of the energy in the magnetosphere.
  - The radiation belts are important because they contain penetrating radiation in a region in which satellites orbit.
  - The plasmasphere contains most of the mass of the magnetosphere.

- The drift of particles with different energies can be critical for the electrodynamics of the inner magnetosphere.

- MHD only includes convection and does not include the energy dependent drift.
The Rice Convection Model
(see Toffoletto et al. Space Science Reviews, 2003)

- Assumes adiabatic (gradient and curvature drift) particle motion in the inner magnetosphere.
- Assume isotropy and average over a flux tube to get the bounced-averaged motion of a particle in a magnetic and electric field

\[
\bar{v}_k(\lambda_k, \bar{x}, t) = \frac{\left[ E - \frac{1}{q_k} \nabla W(\lambda_k, \bar{x}, t) \right] \times \bar{B}(\bar{x}, t)}{B(\bar{x}, t)^2}
\]

where \( W(\lambda_k, \bar{x}, t) \) is the particle kinetic energy, \( q_k \) is the charge and \( \lambda_k \) is the energy invariant defined as

\[
|\lambda_k| = W(\lambda_k, \bar{x}, t) V^{\frac{2}{3}}
\]

which is conserved along a drift path. The subscript \( k \) refers to the species of the particle and \( V \) is the flux tube volume.

\[
V \equiv \int_{sh}^{nh} \frac{ds}{B(\bar{x}, t)}
\]
The Rice Convection Model 2

- Wolf (1983) showed that \( \eta_k(\bar{x},t) \) which is the number of particles per unit magnetic flux follows

\[
\left( \frac{\partial}{\partial t} + \vec{v}_k(\lambda_k, \bar{x}, t) \cdot \nabla \right) \eta_k = S(\eta_k) - L(\eta_k)
\]

where \( S(\eta_k) \) and \( L(\eta_k) \) are source and loss terms.

- The flux tube content is related to the pressure by

\[
PV^\frac{5}{3} = \frac{2}{3} \sum_k \eta_k |\lambda_k|
\]

- The flux tube content is related to the distribution function \( f_k(\lambda_k) \) by

\[
\eta_k = \frac{4\pi \sqrt{2}}{m_k^{\frac{3}{2}}} \int_{\lambda_{\min}}^{\lambda_{\max}} \sqrt{\lambda} f_k(\lambda) d\lambda
\]

where \( \lambda_{\max} - \lambda_{\min} \) is the width of the invariant channel.
The Rice Convection Model 3-The Electric Field

• The electric field is given by

\[ \vec{E} = -\nabla \Phi - \vec{v}_{\text{inductive}} \times \vec{B} \]

where the inductive component comes from changes in the magnetic field and are included implicitly through the time dependent magnetic field.

• The potential is \( \Phi = \Phi_i + \Phi_{\text{corotate}} + \Phi_{||} \)

  – In the simplest case the cross magnetosphere electric field is given by \( \Phi_i = -E_0 y \)

  – The Volland-Stern model (Volland 1973; Stern 1975) is a common variant which includes the effects of shielding of the inner magnetosphere electric field \( \Phi_i = -A_0 yr \)
The Rice Convection Model 4-The Electric Field

- The coupling of the electric field to the ionosphere is via field aligned currents. In force balance \( \vec{J}_\perp = \frac{\vec{B} \times \nabla P}{B^2} \). By using current continuity (\( \nabla \cdot \vec{J} = 0 \)) and integrating over a flux tube one obtains

\[
\frac{j_{nh} - j_{sh}}{B_i} = \frac{\vec{b}}{B} \cdot \nabla V \times \nabla p
\]

This is the equation that gives the region 2 field aligned currents it was originally derived by Vasyliunas (1970).
Simulation Results Showing Pressure in Equatorial Plane

\[ \frac{\dot{j}_{\parallel_{nh}} - \dot{j}_{\parallel_{sh}}}{B_i} = \frac{\vec{b}}{B} \cdot \nabla V \times \nabla p \]
The Rice Convection Model 4 - The Electric Field

• The equation for the parallel currents can be recast in terms of the variables in the RCM

\[
\frac{j_{nh} - j_{sh}}{B_i} = \frac{\hat{b}}{B} \sum \nabla \eta_k (\bar{x}, t) \times \nabla W(\lambda_k, \bar{x}, t)
\]

• Current continuity gives the ionospheric potential

\[
\nabla_i \cdot \left[ \Sigma \cdot (\nabla_i \Phi_i) \right] = - (j_{nh} - j_{sh}) \sin(I)
\]

where \(\Sigma\) is the field-line integrated conductivity tensor, \(I\) is the dip angle of the magnetic field and \(j_{nh} - j_{sh}\) is the ionospheric field aligned current density.

• In addition to the magnetospheric currents the RCM includes the equatorial electrojet to set the low latitude boundary condition.
The Rice Convection Model 5 - The Electric Field

- The high-latitude boundary condition is a Dirichlet boundary where the solar wind potential is specified as a function of local time.

- The transformation of $\Phi_i$ to $\Phi$ occurs by taking into account corotation which transforms the calculation to a coordinate system that doesn’t rotate.

\[
\Phi_{\text{corotate}} = -\frac{\omega E B_M R_E^3}{r} \approx -(89,000\,\text{volts}) \frac{R_E}{r}
\]

- The RCM presently does not included field-aligned electric fields.
## Mapping from MHD to RCM

Comparison of equations of ideal MHD with those used in the RCM

<table>
<thead>
<tr>
<th>Ideal MHD</th>
<th>RCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$</td>
<td>$(\frac{\partial}{\partial t} + \tilde{\vec{v}}_k(\lambda_k, \tilde{x}, t) \cdot \nabla) \eta_k = S(\eta_k) - L(\eta_k)$</td>
</tr>
<tr>
<td>$(\frac{\partial}{\partial t} + \tilde{\vec{v}}_\cdot \nabla)(\rho \tilde{\vec{v}}) = \tilde{j} \times \tilde{\vec{B}} - \nabla P$</td>
<td>$\tilde{j}_k \times \tilde{\vec{B}} = \nabla P_k$</td>
</tr>
<tr>
<td>$(\frac{\partial}{\partial t} + \tilde{\vec{v}} \cdot \nabla)(P \rho^{-5/3}) = 0$</td>
<td>$P = \frac{2}{3} \sum_k \eta_k</td>
</tr>
<tr>
<td>$\nabla \cdot \tilde{\vec{B}} = 0$</td>
<td>Part of the magnetic field model.</td>
</tr>
<tr>
<td>$\nabla \times \tilde{\vec{B}} = \mu_0 \tilde{j}$</td>
<td>Included in magnetic field, but $\tilde{j} \neq \sum_k \tilde{j}_k$.</td>
</tr>
<tr>
<td>$\nabla \times \tilde{\vec{E}} = -\frac{\partial \tilde{\vec{B}}}{\partial t}$</td>
<td>Included implicitly in mapping.</td>
</tr>
<tr>
<td>$\tilde{\vec{E}} + \tilde{\vec{v}} \times \tilde{\vec{B}} = 0$</td>
<td>$\tilde{\vec{E}} \cdot \tilde{\vec{B}} = 0$ and $\tilde{\vec{E}}_\perp + \tilde{\vec{v}}_k \times \tilde{\vec{B}} = \frac{\nabla W(\lambda_k, \tilde{x}, t)}{q_k}$</td>
</tr>
</tbody>
</table>
1. The particles are advanced by using
$$\left(\frac{d}{dt} + \vec{v}_k(\lambda_k, \vec{x}, t) \cdot \nabla\right) \eta_k = S(\eta_k) - L(\eta_k)$$
with
$$\vec{v}_k(\lambda_k, \vec{x}, t) = \frac{[\vec{E} - \frac{1}{q_k} \nabla W(\lambda_k, \vec{x}, t)] \times \vec{B}(\vec{x}, t)}{B(\vec{x}, t)^2}$$

2. The ionospheric electric field is calculated from
$$\nabla_i \cdot [\tilde{\Sigma} \cdot (\nabla_i \Phi_i)] = -\left(j_{\parallel nh} - j_{\parallel sh}\right) \sin(I)$$

3. The magnetospheric electric field is found by mapping along magnetic field lines.

4. Only charge exchange loses are included.

5. Electron precipitation is 30% of strong pitch angle limit.
The Rice Convection Model 7

• The average energy and flux of precipitating electrons are computed from the distribution of plasma sheet electrons.

• The auroral conductances are estimated by using the Robinson et al., (1987) empirical values.

• In one time step: Particles are moved using the computed electric field plus gradient and curvature drift, the new distribution of particles is used to compute field-aligned currents which in turn are used to calculate the electric potential.
Digression on Ionospheric Conductance Models

- **Solar EUV ionization**
  - Empirical model- [Moen and Brekke, 1993]

- **Diffuse auroral precipitation**
  - Thirty percent of strong pitch angle scattering at the inner boundary of the simulation (2-3R_E).
  
  \[ F_E = n_e \left( \frac{kT_e}{2\pi m_e} \right)^{1/2} \]  
  \[ E_0 = kT_e \]  

- **Electron precipitation associated with upward field-aligned currents.**
  
  \[ F_E = \Delta \Phi \left| j_\parallel \right| \]  
  \[ E_0 = e \Delta \Phi \]  
  \[ \Delta \Phi_\parallel = \frac{e^2 n_e}{\sqrt{2\pi m_e kT_e}} \min(0, j_\parallel) \]  
  [Knight, 1972, Lyons et al., 1972]

- **Conductance**
  
  \[ \Sigma_P = n_e \left[ 40 \frac{E_0}{(16 + E_0^2)} \right] F_E^{1/2} \]  
  \[ \Sigma_H = 0.45 E_0^{5/8} \sum_p \]  
  [Robinson et al., 1987]
The Field-Aligned Currents

- (top) Input to the RCM – cross polar cap potential (dark)
- (top) Input Dst.
- (bottom) Circle gives outer boundary.
- (bottom) Field-aligned currents from RCM mapped to equator.
- Pressure gradients associated with the inner edge of the plasma sheet form the region 2 currents.
- These gradients give an E-field opposite to dawn-dusk reduce the E-field in the inner magnetosphere.
The Electric Potential

- The E-field is reduced in the inner magnetosphere.

- If outer E-field increases the inner magnetosphere sees the convection E-field.

- With time the R2 currents increase and a new equilibrium is formed with shielding closer to the Earth.
The RCM Electric Field after an Increase in the Solar Wind Electric Field

Before

After
A Decrease in the Solar Wind E-field Leads to Overshielding

- If the potential difference decreases the R2 currents are too strong leading to overshielding of the inner magnetosphere.

- These E-fields can influence the shape of the plasmapause.
The Asymmetry of the Ring Current

- IMAGE satellite observations show that during storms the ring current maximizes close to midnight.
- The RCM can be used to follow ring current pressure.
- This figure shows the ring current distribution prior to a storm.
The Ring Current Evolution During a Storm

• (left) Quiet time very little ring current.

• (middle) During the main phase the pressure is larger and asymmetric (peak close to midnight and strong dawn-dusk asymmetry).

• (right) During the recovery phase symmetrizes – due to lack of fresh injection, trapping and charge exchange loss.
The Fok Ring Current Model  
(Fok and Moore, 1997) Guiding Center Particle Trajectories

• It is sometimes convenient to express a magnetic field in Euler potential coordinates ($\alpha$ and $\beta$).

$$\vec{A} = \alpha \nabla \beta$$

$$\vec{B} = \nabla \alpha \times \nabla \beta$$

where $\vec{A}$ is the vector potential and $\vec{B}$ is the magnetic field. $\alpha$ and $\beta$ are constant along a field line.

• Northrup [1963] showed that that the bounce-average drift velocity of a charged particle in a magnetic field can be represented by the velocities

$$\langle \dot{\alpha} \rangle = -\frac{1}{q} \frac{\partial H}{\partial \beta}, \langle \dot{\beta} \rangle = \frac{1}{q} \frac{\partial H}{\partial \alpha}$$

where $H$ is the Hamiltonian and $q$ is the charge.
Bounce-Average Guiding Center Trajectories

\[ H = \sqrt{p^2 c^2 + m_0^2 c^4} + q\Phi + q\alpha \partial \beta / \partial t \]

where \( p \) is the momentum, \( c \) the speed of light, \( m_0 \) the rest mass and \( \Phi \) is the cross tail potential.

- Usually particles are identified by their equatorial crossing point. Fok identifies particles by their Earth intercept. Near Earth field lines are dipolar and constant.
- Define \( \xi = B/|\nabla C_1 \times \nabla C_2| \) where \( C_1 \) and \( C_2 \) are general coordinates such that field lines are the intersection of two families of surfaces given by \( C_1 = \text{constant} \) and \( C_2 = \text{constant} \). Northrop showed that by letting \( \beta = C_2 \) gives \( \alpha = \int \xi dC_1 \)
- Fok took \( C_1 = \lambda_i \) and \( C_2 = \phi_i \) where \( \lambda_i \) and \( \phi_i \) are the magnetic latitude and local time.
Apply to the Ring Current

• For the Earth  \( \xi = M_E \sin 2\lambda_i / r_i \), \( \alpha = -M_E \cos 2\lambda_i / 2r_i \)
  
  \( M_E \) is Earth’s dipole moment and \( r_i \) is the distance to the ionosphere.

• Assume the rotation axis is aligned with the magnetic axis the last term of \( H \) becomes

  \[
  q\alpha \frac{\partial \beta}{\partial t} = q\alpha \frac{\partial \phi_i}{\partial t} = q \alpha \Omega
  \]

• This gives \( H = \sqrt{p^2 c^2 + m_0^2 c^4} + q\Phi + q \alpha \Omega \)

• The three terms correspond to the gradient-curvature drift, the electric drift due to the cross-tail \( E \) field and corotation.

• The compression and expansion of the magnetosphere during substorms do not yield \( \partial \phi_i / \partial t \) since the ionospheric point is fixed.

• The substorm induced \( E \) and resulting drift are treated implicitly by the continuously changing gradient and curvature drifts according to change in \( B \).
Variation of Ring Current Species

- The bounce average drift becomes

\[
\langle \dot{\lambda}_i \rangle = -\frac{1}{q\xi} \frac{\partial H}{\partial \phi_i}, \quad \langle \phi \rangle = \frac{1}{q\xi} \frac{\partial H}{\partial \lambda_i}
\]

- Using the equation for \( H \) and the above relationships for the motion the bounce-averaged drift can be calculated if the change in momentum is given.

- Fok characterizes the particles by their adiabatic invariants: \( M \) (magnetic moment) and \( K \) (bounce invariant).

\[
K = J\sqrt{8m_0M} = \sqrt{\int_{s_m}^{s_m} (B_m - B) \, ds}
\]

- Calculating \( B_M \) requires us to actually trace field lines and carry out the integration.
A Bounce-averaged Boltzman Transport Equation

• Once we know $B_M$

\[ M = \frac{p_\perp^2}{2m_0} = \frac{p^2}{2m_0 B_m} \]

\[ p^2(\lambda_i, \phi_i, M, k) = 2m_0 B_m M \]

• Knowing the bounce-averaged drift, the temporal variation of the ring current species can be calculated by solving:

\[
\frac{\partial f}{\partial t} + \left\langle \dot{\lambda} \right\rangle \frac{\partial \bar{f}_s}{\partial \lambda_i} + \left\langle \dot{\phi}_i \right\rangle \frac{\partial \bar{f}_s}{\partial \phi_i} = -v\sigma_s \left\langle n_H \right\rangle \bar{f}_s - \left( \frac{\bar{f}_s}{0.5\tau_b} \right)_{loss \ cone}
\]

where $\bar{f}_s = \bar{f}_s(t, \lambda_i, \phi_i, M, K)$ is the average distribution function between mirror points. $\sigma_s$ is the charge exchange cross section with neutral H and $n_H$ is the hydrogen density. $T_b$ is the bounce period.
Losses

\[
\frac{\partial f}{\partial t} + \left< \lambda \right> \frac{\partial \bar{f}_s}{\partial \lambda_i} + \left< \phi_i \right> \frac{\partial \bar{f}_s}{\partial \phi_i} = -\nu \sigma_s \left< n_H \right> \bar{f}_s - \left( \frac{\bar{f}_s}{0.5 \tau_b} \right)_{loss \ cone}
\]

- The second term on the right is applied only to particles in the loss cone – i.e. particles that mirror below 100km altitude.
- The Fok ring current model only includes losses due to precipitation and charge exchange.
Ring Current Properties Using the Fok Model

- Top equatorial H\(^+\) fluxes during model substorm.
- Bottom precipitating H\(^+\) fluxes.
- Red 1-5 keV
- Green 5-40 keV
- Blue 40-300 keV
- The color bars give the flux range.
- Levels give a range of activity.
- This model used Tsyganenko model B.