ESS 200C - Space Plasma Physics

Winter Quarter 2008

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Schedule of Classes

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ESS 200C – Space Plasma Physics

• There will be two examinations and homework assignments.
• The grade will be based on
  – 35% Exam 1
  – 35% Exam 2
  – 30% Homework

• References
Space Plasma Physics

• Space physics is concerned with the interaction of charged particles with electric and magnetic fields in space.

• Space physics involves the interaction between the Sun, the solar wind, the magnetosphere and the ionosphere.

• Space physics started with observations of the aurorae.
  – Old Testament references to auroras.
  – Greek literature speaks of “moving accumulations of burning clouds”
  – Chinese literature has references to auroras prior to 2000BC
• Aurora over Los Angeles (courtesy V. Peroomian)
Galileo theorized that aurora is caused by air rising out of the Earth’s shadow to where it could be illuminated by sunlight. (Note he also coined the name *aurora borealis* meaning “northern dawn”.)

Descartes thought they are reflections from ice crystals.

Halley suggested that auroral phenomena are ordered by the Earth’s magnetic field.

In 1731 the French philosopher de Mairan suggested they are connected to the solar atmosphere.
• By the 11th century the Chinese had learned that a magnetic needle points north-south.
• By the 12th century the European records mention the compass.
• That there was a difference between magnetic north and the direction of the compass needle (declination) was known by the 16th century.
• William Gilbert (1600) realized that the field was dipolar.
• In 1698 Edmund Halley organized the first scientific expedition to map the field in the Atlantic Ocean.
The Plasma State

• A plasma is an electrically neutral ionized gas.
  – The Sun is a plasma
  – The space between the Sun and the Earth is “filled” with a plasma.
  – The Earth is surrounded by a plasma.
  – A stroke of lightning forms a plasma
  – Over 99% of the Universe is a plasma.

• Although neutral a plasma is composed of charged particles - electric and magnetic forces are critical for understanding plasmas.
The Motion of Charged Particles

- Equation of motion

\[ m \frac{d\vec{v}}{dt} = q\vec{E} + q\vec{v} \times \vec{B} + \vec{F}_g \]

- SI Units
  - mass (m) - kg
  - length (l) - m
  - time (t) - s
  - electric field (E) - V/m
  - magnetic field (B) - T
  - velocity (v) - m/s
  - \( F_g \) stands for non-electromagnetic forces (e.g. gravity) - usually ignorable.
• B acts to change the motion of a charged particle only in directions perpendicular to the motion.
  – Set \( E = 0 \), assume \( B \) along \( z \)-direction.

\[
\begin{align*}
    m\dot{v}_x &= qv_y B \\
    m\dot{v}_y &= -qv_x B \\
    \ddot{v}_x &= \frac{q\dot{v}_y B}{m} = -\frac{q^2 v_x B^2}{m^2} \\
    \ddot{v}_y &= -\frac{q^2 v_y B^2}{m^2}
\end{align*}
\]

– Equations of circular motion with angular frequency (cyclotron frequency or gyro frequency)

\[
\Omega_c = \frac{qB}{m}
\]

– If \( q \) is positive particle gyrates in left handed sense
– If \( q \) is negative particle gyrates in a right handed sense
• Radius of circle (\( r_c \)) - cyclotron radius or Larmor radius or gyro radius. 

\[ v_\perp = \rho_c \Omega_c \]

\[ \rho_c = \frac{mv_\perp}{qB} \]

– The gyro radius is a function of energy.
– Energy of charged particles is usually given in electron volts (eV).
– Energy that a particle with the charge of an electron gets in falling through a potential drop of 1 Volt - 1 eV = 1.6 \times 10^{-19} Joules (J).

• Energies in space plasmas go from electron Volts to kiloelectron Volts (1 keV = 10^3 eV) to millions of electron Volts (1 meV = 10^6 eV).
• Cosmic ray energies go to gigaelectron Volts (1 geV = 10^9 eV).

• The circular motion does no work on a particle

\[ \vec{F} \cdot \vec{v} = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{d(\frac{1}{2}mv^2)}{dt} = q\vec{v} \cdot (\vec{v} \times \vec{B}) = 0 \]

Only the electric field can energize particles!
• The electric field can modify the particles motion.
  – Assume $\vec{E} \neq 0$ but $\vec{B}$ still uniform and $F_g=0$.
  – Frequently in space physics it is ok to set $\vec{E} \cdot \vec{B} = 0$
    • Only $\vec{E}$ can accelerate particles along $\vec{B}$
    • Positive particles go along $E_\parallel$ and negative particles go along $-E_\parallel$
    • Eventually charge separation wipes out $E_\parallel$
  – $E_\perp$ has a major effect on motion.
    • As a particle gyrates it moves along $\vec{E}$ and gains energy
    • Later in the circle it losses energy.
    • This causes different parts of the “circle” to have different radii - it doesn’t close on itself.

\[
\vec{u}_E = \frac{\vec{E} \times \vec{B}}{B^2}
\]

• Drift velocity is perpendicular to $\vec{E}$ and $\vec{B}$
• No charge dependence, therefore no currents
Accelerated by the E field and thus the gyroradius is larger on this part of the orbit
• Any force capable of accelerating and decelerating charged particles can cause them to drift.

\[ \vec{u}_F = \frac{\vec{F} \times \vec{B}}{qB^2} \]

– If the force is charge independent the drift motion will depend on the sign of the charge and can form perpendicular currents.

• Changing magnetic fields cause a drift velocity.

– If \( \vec{B} \) changes over a gyro-orbit the radius of curvature will change.

– \( \rho_c = \frac{mv\perp}{qB} \) gets smaller when the particle goes into a region of stronger B. Thus the drift is opposite to that of \( \vec{E} \times \vec{B} \) motion.

\[ \vec{u}_g = -\frac{1}{2} mv\perp^2 \frac{\nabla B \times \vec{B}}{qB^3} = \frac{1}{2} mv\perp^2 \frac{\vec{B} \times \nabla B}{qB^3} \]

– \( u_g \) depends on the charge so it can yield perpendicular currents.
• The change in the direction of the magnetic field along a field line can cause motion.
  – The curvature of the magnetic field line introduces a drift motion.
    • As particles move along the field they undergo centrifugal acceleration.
      \[ \vec{F} = \frac{m v^2}{R_c} \hat{R}_c \]
    • \( R_c \) is the radius of curvature of a field line \( \left( \frac{\hat{n}}{R_c} = -(\hat{b} \cdot \nabla)\hat{b} \right) \)
      where \( \hat{b} = \frac{\vec{B}}{B} \), \( \hat{n} \) is perpendicular to \( \vec{B} \) and points away from the center of curvature, \( \vec{v}_\parallel \) is the component of velocity along \( \vec{B} \)
      \[ \vec{u}_c = \frac{m v^2 \vec{B} \times (\hat{b} \cdot \nabla)\hat{b}}{qB^2} = -\frac{m v^2 \vec{B} \times \hat{n}}{R_c qB^2} \]
    • Curvature drift can cause currents.
• The Concept of the Guiding Center
  
  Separates the motion (\( \mathbf{v} \)) of a particle into motion perpendicular (\( \mathbf{v}_\perp \)) and parallel (\( \mathbf{v}_\parallel \)) to the magnetic field.
  
  To a good approximation the perpendicular motion can consist of a drift (\( \mathbf{v}_D \)) and the gyro-motion (\( \mathbf{v}_{\Omega_c} \))

\[
\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp = \mathbf{v}_\parallel + \mathbf{v}_D + \mathbf{v}_{\Omega_c} = \mathbf{v}_{gc} + \mathbf{v}_{\Omega_c}
\]
  
  Over long times the gyro-motion is averaged out and the particle motion can be described by the guiding center motion consisting of the parallel motion and drift. This is very useful for distances \( l \) such that \( \rho_c/l << 1 \) and time scales \( \tau \) such that \( (\Omega\tau)^{-1} << 1 \)
• Maxwell’s equations
  – Poisson’s Equation

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]

• \( \vec{E} \) is the electric field
• \( \rho \) is the charge density
• \( \varepsilon_0 \) is the electric permittivity (8.85 X 10^{-12} Farad/m)
  – Gauss’ Law (absence of magnetic monopoles)

\[ \nabla \cdot \vec{B} = 0 \]

• \( \vec{B} \) is the magnetic field
Faraday’s Law

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Ampere’s Law

\[ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \]

- \( c \) is the speed of light.
- \( \mu_0 \) is the permeability of free space, \( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \)
- \( \vec{J} \) is the current density
Maxwell’s equations in integral form

\[ \oint_A \vec{E} \cdot \hat{n} dA = \frac{1}{\varepsilon_0} \int \rho dV \]

- A is the area, dA is the differential element of area
- \( \hat{n} \) is a unit normal vector to dA pointing outward.
- V is the volume, dV is the differential volume element

\[ \oint_A \vec{B} \cdot \hat{n} dA = 0 \]

\[ \oint_c \vec{E} \cdot d\vec{s} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}' dF = -\frac{\partial \Phi}{\partial t} \]

- \( \hat{n}' \) is a unit normal vector to the surface element dF in the direction given by the right hand rule for integration around C, and \( \Phi \) is magnetic flux through the surface.
- d\( \vec{s} \) is the differential element around C.

\[ \oint_c \vec{B} \cdot d\vec{s} = \frac{1}{c^2} \int \frac{\partial \vec{E}}{\partial t} \cdot \hat{n}' dF + \mu_0 \oint J \cdot \hat{n} dF \]
The first adiabatic invariant

\[ \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \]

says that changing \( \vec{B} \) drives \( \vec{E} \) (electromotive force). This means that the particles change energy in changing magnetic fields.

Even if the energy changes there is a quantity that remains constant provided the magnetic field changes slowly enough. \[ \mu = \frac{1}{2} \frac{mv^2}{B} = \text{const}. \]

\( \mu \) is called the magnetic moment. In a wire loop the magnetic moment is the current through the loop times the area.

As a particle moves to a region of stronger (weaker) \( B \) it is accelerated (decelerated).
• For a coordinate in which the motion is periodic the action integral
  \[ J_i = \int p_i dq_i = \text{constant} \]
is conserved. Here \( p_i \) is the canonical momentum (\( \vec{p} = m\dot{\vec{v}} + q\vec{A} \) where \( \vec{A} \) is the vector potential).

• First term \( \int m\nu_\perp ds = 2\pi \rho m\nu_\perp = 4\pi m\mu/\rho \)

• Second term \( \int q\vec{A} \cdot d\vec{S} = \int q\nabla \times \vec{A} \cdot d\vec{S} = \int q\vec{B} \cdot d\vec{S} = -\frac{2\pi m^2\nu^2}{qB} = -2\pi m\mu/\rho \)

• For a gyrating particle
  \[ J_1 = \int \vec{p}_\perp \cdot d\vec{S} = \frac{2\pi m}{q} \mu \]

• The action integrals are conserved when the properties of the system change slowly compared to the period of the coordinate.
• The magnetic mirror
  – As a particle gyrates the current will be

\[ I = \frac{q}{T_c} \quad \text{where} \quad T_c = \frac{2\pi}{\Omega_c} \]

\[ A = \pi r_c^2 = \pi \frac{v_\perp^2}{\Omega_c^2} \]

\[ IA = \frac{\pi v_\perp^2 |q| \Omega_c}{2\pi \Omega_c^2} = \frac{mv_\perp^2}{2B} = \mu \]

– The force on a dipole magnetic moment is

\[ \vec{F} = -\vec{\mu} \cdot \nabla B = -\mu \frac{dB}{dz} \]

where \( \vec{\mu} = \mu \hat{b} \)
• The force is along \( \vec{B} \) and away from the direction of increasing \( B \).
• Since \( E_\parallel = 0 \) and kinetic energy must be conserved
  \[ \frac{1}{2} mv^2 = \frac{1}{2} m(v_\parallel^2 + v_\perp^2) \]
a decrease in \( v_\parallel \) must yield an increase in \( v_\perp \)
• Particles will turn around when
  \[ B = \frac{1}{2} mv^2 / \mu \]
• The second adiabatic invariant
  
  – The integral of the parallel momentum over one complete bounce between mirrors is constant (as long as B doesn’t change much in a bounce).

  \[ J = \int_{s_1}^{s_2} 2mv \parallel \, ds = \text{const}. \]

  – Using conservation of energy and the first adiabatic invariant

  \[ J = \int_{s_1}^{s_2} 2mv(1 - \frac{B}{B_m})^{\frac{1}{2}} \, ds = \text{const}. \]

  here \( B_m \) is the magnetic field at the mirror point.
As particles bounce they will drift because of gradient and curvature drift motion.

If the field is a dipole their trajectories will take them around the planet and close on themselves.

**The third adiabatic invariant**

As long as the magnetic field doesn’t change much in the time required to drift around a planet the magnetic flux \( \Phi = \int \vec{B} \cdot \hat{n} dA \) inside the orbit must be constant.

Note it is the total flux that is conserved including the flux within the planet.
Gyro Motion

Bounce Motion

Drift Motion

$V_g$

$V_b$

$V_d$
• Limitations on the invariants
  – \( \mu \) is constant when there is little change in the field’s strength over a cyclotron path.
    \[
    \left| \frac{\nabla B}{B} \right| \ll \frac{1}{\rho_c}
    \]
  – All invariants require that the magnetic field not change much in the time required for one cycle of motion
    \[
    \left| \frac{1}{B} \frac{\partial B}{\partial t} \right| \ll \frac{1}{\tau}
    \]
    where \( \tau \) is the orbit period.
    \[
    \tau_\mu \sim 10^{-6} - 10^{-3} \text{ s}
    \]
    \[
    \tau_j \sim 1 \text{ s}
    \]
    \[
    \tau_\Phi \sim m
    \]
The Properties of a Plasma

- A plasma as a collection of particles
  - The properties of a collection of particles can be described by specifying how many there are in a 6 dimensional volume called phase space.
    - There are 3 dimensions in “real” or configuration space and 3 dimensions in velocity space.
    - The volume in phase space is \( dvdr = dv_x dv_y dv_z dx dy dz \)
    - The number of particles in a phase space volume is \( f(\vec{r}, \vec{v}, t) dvdr \)
      where \( f \) is called the distribution function.
  - The density of particles of species “s” (number per unit volume)
    \[
    n_s(\vec{r}, t) = \int f_s(\vec{r}, \vec{v}, t) dv
    \]
  - The average velocity (bulk flow velocity)
    \[
    \bar{u}_s(\vec{r}, t) = \frac{\int \vec{v} f_s(\vec{r}, \vec{v}, t) dv}{\int f_s(\vec{r}, \vec{v}, t) dv}
    \]
– Average random energy
\[
\left\langle \frac{1}{2} m_s (\vec{v} - \vec{u}_s)^2 \right\rangle = \int \frac{1}{2} m_s (\vec{v} - \vec{u}_s)^2 f_s (\vec{r}, \vec{v}, t) dv / \int f_s (\vec{r}, \vec{v}, t) dv
\]

– The partial pressure of s is given by
\[
\frac{p_s}{n_s} = \left( \frac{2}{N} \left\langle \frac{1}{2} m_s (\vec{v} - \vec{u}_s)^2 \right\rangle \right)
\]

where N is the number of independent velocity components (usually 3).

– In equilibrium the phase space distribution is a Maxwellian distribution
\[
f_s (\vec{r}, \vec{v}) = A_s \exp \left[ \frac{1}{2} m_s (\vec{v} - \vec{u}_s)^2 \right] \frac{1}{kT_s}
\]

where \( A_s = n_s \left( m/2\pi kT \right)^{\frac{3}{2}} \)
• For monatomic particles in equilibrium

\[ \langle \frac{1}{2} m_s (\vec{v} - \vec{u}_s)^2 \rangle = NkT / 2 \]

\[ p_s = n_s kT_s \]

where \( k \) is the Boltzmann constant (\( k=1.38 \times 10^{-23} \text{ JK}^{-1} \))

• For monatomic particles in equilibrium

\[ \langle \frac{1}{2} \left( m_s (\vec{v} - \vec{u}_s)^2 \right) \rangle = NkT_s / 2 \]

• This is true even for magnetized particles.

• The ideal gas law becomes

\[ p_s = n_s kT_s \]
Other frequently used distribution functions.

- The bi-Maxwellian distribution

\[ f_s(r, v) = A'_s \exp \left[ -\frac{1}{2} m_s \left( v_\parallel - u_\parallel s \right)^2 \right] \exp \left[ -\frac{1}{2} m_s \left( v_\perp - u_\perp s \right)^2 \right] \]

- where \( A'_s = A_s \frac{T_s^3}{\left( T_\perp T_\parallel \right)^{\frac{3}{2}}} \)
- It is useful when there is a difference between the distributions perpendicular and parallel to the magnetic field

- The kappa distribution

\[ f_s(r, v) = A_{\kappa s} \left[ 1 + \frac{1}{2} m_s \left( \vec{v} - \vec{u}_s \right)^2 \right]^{-\kappa-1} \]

- \( \kappa \) characterizes the departure from Maxwellian form.
- \( E_{Ts} \) is an energy.
- At high energies \( E >> \kappa E_{Ts} \) it falls off more slowly than a Maxwellian (similar to a power law)
- For \( \kappa \rightarrow \infty \) it becomes a Maxwellian with temperature \( kT = E_{Ts} \)
What makes an ionized gas a plasma?

- The electrostatic potential of an isolated ion \( \phi = \frac{q}{4\pi\varepsilon_0r} \).
- The electrons in the gas will be attracted to the ion and will reduce the potential at large distances.
- If we assume neutrality Poisson’s equation around a test charge \( q_0 \) is

\[
\nabla^2 \phi(\vec{r}) = -\frac{\rho}{\varepsilon_0} = -\frac{q_0}{\varepsilon_0} \delta^3(\vec{r}) + \frac{en_0}{\varepsilon_0} \left[ \exp\left(\frac{e\phi}{kT_e}\right) - \exp\left(\frac{-e\phi}{kT_{ion}}\right) \right]
\]

- Expanding in a Taylor series for \( r>0 \) and \( \left|\frac{e\phi}{kT}\right| \ll 1 \) for both electrons and ions

\[
\nabla^2 \phi(\vec{r}) = \frac{en}{\varepsilon_0} \left[ 1 + \frac{e\phi}{kT_e} - 1 + \frac{e\phi}{kT_{ion}} \right] = \frac{e^2n}{\varepsilon_0 kT} \phi
\]
The Debye length \( \lambda_D \) is

\[
\lambda_D = \left( \frac{\varepsilon_0 kT}{n e^2} \right)^{1/2}
\]

where \( n \) is the electron number density and \( e \) is the electron charge.

The number of particles within a Debye sphere

\[
N_D = \frac{4\pi n \lambda_D^3}{3}
\]

needs to be large for shielding to occur \((N_D >> 1)\). Far from the central charge the electrostatic force is shielded.
The plasma frequency

Consider a slab of plasma of thickness L.

At t=0 displace the electron part of the slab by \( \delta_e \ll L \) and the ion part of the slab by \( \delta_i \ll L \) in the opposite direction.

\[ \delta = \delta_e - \delta_i \]

Poisson’s equation gives

\[ E = \frac{en_0}{\varepsilon_0} \delta \]

The equations of motion for the electron and ion slabs are

\[ m_e \frac{d^2 \delta_e}{dt^2} = -eE \]

\[ m_{\text{ion}} \frac{d^2 \delta_i}{dt^2} = eE \]

\[ \frac{d^2 \delta}{dt^2} = \frac{d^2 \delta_e}{dt^2} - \frac{d^2 \delta_i}{dt^2} = -\left( \frac{e^2 n_0}{\varepsilon_0 m_e} + \frac{e^2 n_0}{\varepsilon_0 m_{\text{ion}}} \right) \delta \]
– The frequency of this oscillation is the plasma frequency

\[ \omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2 \]

\[ \omega_{pe}^2 = \frac{e^2 n_0}{\varepsilon_0 m_e} \]

\[ \omega_{pi}^2 = \frac{e^2 n_0}{\varepsilon_0 m_{\text{ion}}} \]

– Because \( m_{\text{ion}} \gg m_e \), \[ \omega_p \approx \omega_{pe} \]
• A note on conservation laws
  – Consider a quantity that can be moved from place to place.
  – Let \( \vec{f} \) be the flux of this quantity – i.e. if we have an element of area \( \delta \vec{A} \)
    then \( \vec{f} \cdot \delta \vec{A} \) is the amount of the quantity passing the area element per unit time.
  – Consider a volume \( V \) of space, bounded by a surface \( S \).
  – If \( \sigma \) is the density of the substance then the total amount in the volume is \( \int_V \sigma \, dV \)
  – The rate at which material is lost through the surface is \( \oint_S \vec{f} \cdot d\vec{A} \)
    \[
    \frac{d}{dt} \int_V \sigma \, dV = -\oint_S \vec{f} \cdot d\vec{A}
    \]
  – Use Gauss’ theorem
    \[
    \int_V \left\{ \frac{\partial \sigma}{\partial t} + \nabla \cdot \vec{f} \right\} \, dV = 0
    \]
    \[
    \frac{\partial \sigma}{\partial t} = -\nabla \cdot \vec{f}
    \]
  – An equation of the preceeding form means that the quantity whose density is \( \sigma \) is conserved.
Magnetohydrodynamics (MHD)

- The average properties are governed by the basic conservation laws for mass, momentum and energy in a fluid.
- Continuity equation

\[ \frac{\partial n_s}{\partial t} + \nabla \cdot n_s \vec{u}_s = S_s - L_s \]

- \( S_s \) and \( L_s \) represent sources and losses. \( S_s - L_s \) is the net rate at which particles are added or lost per unit volume.
- The number of particles changes only if there are sources and losses.
- \( S_s, L_s, n_s, \) and \( u_s \) can be functions of time and position.
- Assume \( S_s = 0 \) and \( L_s = 0 \), \( \rho_s = m_s n_s \), \( \int \rho_s dr = M_s \) where \( M_s \) is the total mass of \( s \) and \( dr \) is a volume element (e.g. \( dx dy dz \)).

\[ \frac{\partial M_s}{\partial t} + \int \nabla \cdot (\rho_s \vec{u}_s) dr = \frac{\partial M_s}{\partial t} + \int \rho_s \vec{u}_s \cdot d\vec{s} \]

where \( d\vec{s} \) is a surface element bounding the volume.
Momentum equation

\[
\left( \frac{\partial \rho_s \vec{u}_s}{\partial t} + \nabla \cdot \left( \rho_s \vec{u}_s \vec{u}_s \right) \right) = -\nabla p_s + \rho_{qs} \vec{E} + \vec{J}_s \times \vec{B} + \rho_s \vec{F}_g / m_s
\]

\[
\rho_s \left( \frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s \cdot \nabla \vec{u}_s \right) + m_s \vec{u}_s (S_s - L_s) = -\nabla p_s + \rho_{qs} \vec{E} + \vec{J}_s \times \vec{B} + \rho_s \vec{F}_g / m_s
\]

where \( \rho_{qs} = q_s n_s \) is the charge density, \( \vec{J}_s = q_s n_s \vec{u}_s \) is the current density, and the last term is the density of non-electromagnetic forces.

The operator \( \left( \frac{\partial}{\partial t} + \vec{u}_s \cdot \nabla \right) \) is called the convective derivative and gives the total time derivative resulting from intrinsic time changes and spatial motion.

If the fluid is not moving \( (u_s = 0) \) the left side gives the net change in the momentum density of the fluid element.

The right side is the density of forces

- If there is a pressure gradient then the fluid moves toward lower pressure.
- The second and third terms are the electric and magnetic forces.
- The $\vec{u}_s \cdot \nabla \vec{u}_s$ term means that the fluid transports momentum with it.

- **Combine the species for the continuity and momentum equations**
  - Drop the sources and losses, multiply the continuity equations by $m_s$, assume $n_p=n_e$ and add.

  $\begin{array}{c}
  \text{Continuity} \\
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0
  \end{array}$

  - Add the momentum equations and use $m_e<<m_p$

  $\begin{array}{c}
  \text{Momentum} \\
  \rho \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \vec{J} \times \vec{B} + \rho \frac{F_g}{m}
  \end{array}$
• Energy equation

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + U \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + U \right) \vec{u} + p \vec{u} + \vec{q} \right] = \vec{J} \cdot \vec{E} + \rho \vec{u} \cdot \vec{F}_g / m
\]

where \( \vec{q} \) is the heat flux, \( U \) is the internal energy density of the monatomic plasma \( (U = nNkT / 2) \) and \( N \) is the number of degrees of freedom

- \( \vec{q} \) adds three unknowns to our set of equations. It is usually treated by making approximations so it can be handled by the other variables.
- Many treatments make the adiabatic assumption (no change in the entropy of the fluid element) instead of using the energy equation

\[
\frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p = c_s^2 \left( \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho \right)
\]

\[
p \rho^{-\gamma} = \text{const.}
\]

where \( c_s \) is the speed of sound \( c_s^2 = \gamma \ p / \rho \) and \( \gamma = c_p / c_v \)

\( c_p \) and \( c_v \) are the specific heats at constant pressure and constant volume. It is called the polytropic index. In thermodynamic equilibrium \( \gamma = (N + 2) / N = 5 / 3 \)
• Maxwell’s equations

\[
\begin{align*}
\frac{\partial \vec{B}}{\partial t} &= -\nabla \times \vec{E} \\
\nabla \times \vec{B} &= \mu_0 \vec{J}
\end{align*}
\]

- \( \nabla \cdot \vec{B} = 0 \) doesn’t help because

\[
\frac{\partial (\nabla \cdot \vec{B})}{\partial t} = -\nabla \cdot \nabla \times \vec{E} = 0
\]

- There are 14 unknowns in this set of equations - \( \vec{E}, \vec{B}, \vec{J}, \vec{u}, \rho, p \)
- We have 11 equations.

• Ohm’s law

- Multiply the momentum equations for each individual species by \( q_s/m_s \) and subtract.

\[
\vec{J} = \sigma \left\{ (\vec{E} + \vec{u} \times \vec{B}) + \frac{1}{ne} \nabla p_e - \frac{1}{ne} \vec{J} \times \vec{B} - \frac{m_e}{ne^2} \left[ \frac{\partial \vec{J}}{\partial t} + \nabla \cdot (\vec{J} \vec{u}) \right] \right\}
\]

where \( \vec{J} = \sum_s q_s n_s \vec{u}_s \) and \( \sigma \) is the electrical conductivity
- Often the last terms on the right in Ohm’s Law can be dropped

\[ \vec{J} = \sigma (\vec{E} + \vec{u} \times \vec{B}) \]

- If the plasma is collisionless, \( \sigma \) may be very large so

\[ \vec{E} + \vec{u} \times \vec{B} = 0 \]
Frozen in flux

- Combining Faraday’s law \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \), and Ampere’ law \( \nabla \times \vec{B} = \mu_0 \vec{J} \) with \( \vec{J} = \sigma (\vec{E} + \vec{u} \times \vec{B}) \)

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \eta_m \nabla^2 \vec{B}
\]

where \( \eta_m = 1/\sigma \mu_0 \) is the magnetic viscosity

- If the fluid is at rest this becomes a “diffusion” equation

\[
\frac{\partial \vec{B}}{\partial t} = \eta_m \nabla^2 \vec{B}
\]

- The magnetic field will exponentially decay (or diffuse) from a conducting medium in a time \( \tau_D = L_B^2/\eta_m \) where \( L_B \) is the system size.
- On time scales much shorter than $\tau_D$ 
  \[ \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) \]

- The electric field vanishes in the frame moving with the fluid.

- Consider the rate of change of magnetic flux 
  \[ \frac{d\Phi}{dt} = \frac{d}{dt} \int_A \vec{B} \cdot \hat{n} dA = \int_A \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} dA + \oint_C \vec{B} \cdot (\vec{u} \times d\vec{l}) \]

- The first term on the right is caused by the temporal changes in $B$

- The second term is caused by motion of the boundary

- The term $\vec{u} \times d\vec{l}$ is the area swept out per unit time

- Use the identity $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{C} \cdot \vec{A} \times \vec{B}$ and Stoke’s theorem 
  \[ \frac{d\Phi}{dt} = \int_A \left( \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) \right) \cdot \hat{n} dA = 0 \]

- If the fluid is initially on surface $s$ as it moves through the system the flux through the surface will remain constant even though the location and shape of the surface change.
• Magnetic pressure and tension

\[ \vec{F}_B = \vec{J} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = -\nabla B^2 / 2\mu_0 + (\vec{B} \cdot \nabla) \vec{B} / \mu_0 \]

since

\[ \vec{A} \times (\nabla \times \vec{B}) = (\nabla \vec{B}) \cdot \vec{A} - (\vec{A} \cdot \nabla) \vec{B} \]

- \[ p_B = B^2 / 2\mu_0 \] A magnetic pressure analogous to the plasma pressure (- \( \nabla p \))

- \[ \beta = \frac{\rho}{B^2 / 2\mu_0} \] A “cold” plasma has \( \beta \ll 1 \) and a “warm” plasma has \( \beta \geq 1 \)

- In equilibrium \( \vec{J} \times \vec{B} = \nabla p \)

• Pressure gradients form currents
– The second term in $\vec{J} \times \vec{B}$ can be written as a sum of two terms

$$\frac{[(\vec{B} \cdot \nabla)\vec{B}]}{\mu_0} = \frac{\hat{b}\vec{B} \cdot \nabla}{\mu_0} + \frac{(B^2)}{\mu_0} \hat{b} \cdot \nabla \hat{b}$$

$\hat{b}\vec{B} \cdot \nabla / \mu_0 = \frac{\hat{b} \vec{B} \cdot \nabla B^2}{2\mu_0}$ cancels the parallel component of the $-\frac{\nabla B^2}{2\mu_0}$ term. Thus only the perpendicular component of the magnetic pressure exerts a force on the plasma.

$-(B^2/\mu_0) \hat{b} \cdot \nabla \hat{b} = -\left( \frac{\hat{n}B^2}{\mu_0 R_C} \right)$ is the magnetic tension and is directed antiparallel to the radius of curvature ($R_C$) of the field line. Note that $\hat{n}$ is directed outward.
Some elementary wave concepts

– For a plane wave propagating in the x-direction with wavelength $\lambda$ and frequency $f$, the oscillating quantities can be taken to be proportional to sines and cosines. For example the pressure in a sound wave propagating along an organ pipe might vary like

$$p = p_0 \sin(kt - \omega t)$$

– A sinusoidal wave can be described by its frequency $\omega$ and wave vector $\vec{k}$. (In the organ pipe the frequency is $f$ and $\omega = 2\pi f$. The wave number is $k = 2\pi/\lambda$). \[ \vec{B}(r,t) = \vec{B}_0 \left( \cos(\vec{k} \cdot \vec{r} - \omega t) + i \sin(\vec{k} \cdot \vec{r} - \omega t) \right) \]

$$\vec{B}(r,t) = \vec{B}_0 \exp \{i(\vec{k} \cdot \vec{r} - \omega t)\}$$
• The exponent gives the phase of the wave. The phase velocity specifies how fast a feature of a monotonic wave is moving.

\[ v_{ph} = \frac{\omega}{k^2} \hat{k} \]

• Information propagates at the group velocity. A wave can carry information provided it is formed from a finite range of frequencies or wave numbers. The group velocity is given by

\[ v_g = \frac{\partial \omega}{\partial k} \]

• The phase and group velocities are calculated and waves are analyzed by determining the dispersion relation

\[ \omega = \omega(k) \]
When the dispersion relation shows asymptotic behavior toward a given frequency, $\omega_{res}$, $v_g$ goes to zero, the wave no longer propagates and all the wave energy goes into stationary oscillations. This is called a resonance.
• MHD waves - natural wave modes of a magnetized fluid
  
  – Sound waves in a fluid
    
    • Longitudinal compressional oscillations which propagate at
      
      \[ c_s = \left( \frac{\partial p}{\partial \rho} \right)^{\frac{1}{2}} = \left( \frac{\gamma p}{\rho} \right)^{\frac{1}{2}} \]

    • \[ c_s = \left[ \gamma \left( \frac{kT}{m} \right) \right]^{\frac{1}{2}} \] and is comparable to the thermal speed.
- Incompressible Alfvén waves
  - Assume $\sigma \to \infty$, incompressible fluid with $\vec{B}_0$ background field and homogeneous

  \[
  \begin{align*}
  \nabla \cdot \vec{u} &= 0 \\
  \text{We want plane wave solutions } b = b(z,t), u = u(z,t), b_z = u_z = 0 \\
  \text{Ampere’s law gives the current} \quad \vec{J} &= -\frac{1}{\mu_0} \frac{\partial b}{\partial z} \hat{i} \\
  \text{Ignore convection} (\vec{u} \cdot \nabla) &= 0 \quad \rho \frac{\partial \vec{u}}{\partial t} = -\nabla p + \vec{J} \times \vec{B}
  \end{align*}
  \]
• Since \( \frac{\partial p}{\partial x} = 0 \), \( J_y = 0 \) and \( J_z = 0 \) the x-component of momentum becomes

\[
\rho \frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + (J_y B_0 - b_y J_z) = 0
\]

\[ \bar{u} = u \hat{j} \]

\[ \bar{E} = -u B_0 \hat{i} \]

• Faraday’s law gives

\[
\frac{\partial b}{\partial t} \hat{j} = -\frac{\partial E_x}{\partial z} \hat{j} = B_0 \frac{\partial u}{\partial z} \hat{j}
\]

• The y-component of the momentum equation becomes

\[
\frac{\partial u_y}{\partial t} = -\frac{1}{\rho} \bar{J} \times \bar{B}_0 = \frac{B_0}{\mu_0 \rho} \frac{\partial b}{\partial z}
\]

• Differentiating Faraday’s law and substituting the y-component of momentum

\[
\frac{\partial^2 b}{\partial t^2} = B_0 \frac{\partial^2 u}{\partial z \partial t} = \left( \frac{B^2}{\mu_0 \rho} \right) \frac{\partial^2 b}{\partial z^2}
\]
where \( C_A = \left( \frac{B^2}{\mu_0 \rho} \right)^{1/2} \) is called the Alfvén velocity.

- The most general solution is \( b = b(z \pm C_A t) \). This is a disturbance propagating along magnetic field lines at the Alfvén velocity.
• Compressible solutions
  – In general incompressibility will not always apply.
  – Usually this is approached by assuming that the system starts in equilibrium and that perturbations are small.
• Assume uniform $B_0$, perfect conductivity with equilibrium pressure $p_0$ and mass density $\rho_0$

\[\rho_T = \rho_0 + \rho\]
\[p_T = p_0 + p\]
\[\vec{B}_T = \vec{B}_0 + \vec{b}\]
\[\vec{u}_T = \vec{u}\]
\[\vec{J}_T = \vec{J}\]
\[\vec{E}_T = \vec{E}\]
- Continuity  \( \frac{\partial \rho}{\partial t} = -\rho_0 (\nabla \cdot \vec{u}) \)

- Momentum  \( \rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p - \frac{1}{\mu_0} (\vec{B}_0 \times (\nabla \times \vec{b})) \)

- Equation of state  \( \nabla p = \left( \frac{\partial p}{\partial \rho} \right)_0 \nabla \rho = C_s^2 \nabla \rho \)

- Differentiate the momentum equation in time, use Faraday’s law and the ideal MHD condition  \( \vec{E} = -\vec{u} \times \vec{B}_0 \)

\[
\frac{\partial \vec{b}}{\partial t} = -(\nabla \times \vec{E}) = \nabla \times (\vec{u} \times \vec{B}_0)
\]

\[
\frac{\partial^2 \vec{u}}{\partial t^2} - C_s^2 \nabla (\nabla \cdot \vec{u}) + \vec{C}_A \times (\nabla \times (\nabla \times (\vec{u} \times \vec{C}_A))) = 0
\]

where \( \vec{C}_A = \frac{\vec{B}}{(\mu_0 \rho)^{\frac{1}{2}}} \)
- For a plane wave solution $\vec{u} \sim \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$

- The dispersion relationship between the frequency ($\omega$) and the propagation vector ($\vec{k}$) becomes

$$-\omega^2 \vec{u} + (C_s^2 + C_A^2)(\vec{k} \cdot \vec{u})\vec{k} + (\vec{C}_A \cdot \vec{k})[(\vec{C}_A \cdot \vec{k})\vec{u} - (\vec{C}_A \cdot \vec{u})\vec{k} - (\vec{k} \cdot \vec{u})\vec{C}_A] = 0$$

This came from replacing derivatives in time and space by

$$\frac{\partial}{\partial t} \rightarrow -i \omega$$

$$\nabla \rightarrow i \vec{k}$$

$$\nabla \cdot \rightarrow i \vec{k} \cdot$$

$$\nabla \times \rightarrow i \vec{k} \times$$

- Case 1 $\vec{k} \perp \vec{B}_0$

$$\omega^2 \vec{u} = (C_s^2 + C_A^2)(\vec{k} \cdot \vec{u})\vec{k}$$
• The fluid velocity must be along $\vec{k}$ and perpendicular to $\vec{B}_0$

\[
\vec{B}_0 \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \longrightarrow \quad \vec{k} \\
\quad \quad \quad \quad \quad \quad \quad \quad \longrightarrow \quad \vec{u}
\]

\[
\nu_{ph} = \vec{k} \left( \frac{\omega}{k} \right) = \pm \left( C_s^2 + C_A^2 \right)^{\frac{1}{2}}
\]

• These are magnetosonic waves

- Case $2\vec{k}||\vec{B}_0$

\[
(k^2 C_A^2 - \omega^2) \vec{u} + \left( \left( \frac{C_s^2}{C_A^2} \right) - 1 \right) k^2 \left( \vec{C}_A \cdot \vec{u} \right) \vec{C}_A = 0
\]

• A longitudinal mode with $\vec{u}||\vec{k}$ with dispersion relationship $\frac{\omega}{k} = \pm C_s$ (sound waves)

• A transverse mode with $\vec{k} \cdot \vec{u} = 0$ and $\frac{\omega}{k} = \pm C_A$ (Alfvén waves)
• Alfven waves propagate parallel to the magnetic field.

• The tension force acts as the restoring force.

• The fluctuating quantities are the electromagnetic field and the current density.
- Arbitrary angle between $\vec{k}$ and $\vec{B}_0$

$V_A = 2C_S$

Phase Velocities