Adiabatic Charged Particle Motion in Rapidly Rotating Magnetospheres

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When the $cE \times B/B^2$ drift velocity is large enough to exceed the particle gyro velocity, adiabatic theory becomes more complicated and less useful: more complicated because there are five drifts in addition to the $E \times B$ gradient, and line curvature drifts and three more terms in the parallel equation of motion; less useful because the second and third invariants $J$ and $\Phi$ are no longer valid in mirror geometry due to the rapid drift across field lines that destroys any semblance of periodicity in the bounce motion. (Approximate periodicity is necessary to have an adiabatic invariant.) But the special case of a rapidly rotating rigid magnetic field, with $E_\parallel = 0$, is an exception. If the field at any time is merely a rotation of that at an earlier time, the drift and parallel equations simplify, and even better, the second invariant is again valid. The drift equation now has two rather than five additional terms—a Coriolis drift and a centrifugal drift. Just as in a slowly or nonrotating mirror system, the second invariant now provides a (rapidly rotating) drift shell to which the guiding center is confined. The particle kinetic energy in the rotating frame, minus the centrifugal potential, is an exact constant of motion in a rigid rotator. This constant is well known but does not by itself put limits on radial motion toward or away from the rotation axis, hence it does not limit particle energy changes. But the drift shell does limit and make periodic any radial excursions, and this is why previous studies of particle motion, in particular examples of rigid rotators, have shown that particles experience no steady energy gain or loss.

1. Introduction

The theory of adiabatic particle motion in electromagnetic fields is well developed through first order in gyroradius [e.g., Alfvén, 1950; Northrop, 1963; Roederer, 1970] and to a lesser extent through second order [Northrop and Rome, 1978]. Adiabatic theory gives the equations of motion of the guiding center and the approximate constants (magnetic moment $M$, longitudinal invariant $J$, and flux invariant $\Phi$) of that motion. These invariants place strong constraints on the particle motion in mirror geometry. Furthermore, the invariance of $M$ and $J$ is crucial in deriving diffusion equations for trapped particles in planetary magnetospheres [Birmingham et al., 1967].

All proofs of the invariance of $J$ require that the magnitude of the electric field drift velocity $v_E = cE \times B/B^2$ be small compared to the particle gyro velocity $v_\phi$. In this situation, the usual three guiding center drifts ($u_B$, magnetic field gradient, and field line curvature) are the only large ones. However, when $v_E \gtrsim v_\phi$ the other five drifts besides the curvature drift that arise from the guiding center acceleration become comparable to the curvature (and gradient) drifts and must be included [Northrop, 1963, equations 1.13 and 1.17]. Furthermore, when $u_E \gtrsim v_\phi$, $J$ and $\Phi$ are generally no longer adiabatic invariants, and the usual diffusion equation cannot be justified. The reason that $J$ is no longer conserved is that the guiding center velocity component perpendicular to the magnetic field $B$ is comparable to or greater than the component parallel to $B$, and after one mirror bounce period the guiding center does not return anywhere near the mirror point it left a period earlier. In direct proofs of the invariance of $J$ [Northrop and Teller, 1960; Northrop, 1963] one shows directly that the drifts conserve $J$, assuming the presence of only the gradient, curvature, and $u_E$ drifts. If one has present in addition the other five drifts, the proof cannot be carried through. The same difficulty appears in different guise if one tries to carry out those proofs that employ Hamiltonian theory [Kruskal, 1962].

Because the invariants are such powerful tools, it is important to know whether the invariance of $J$ (and $\Phi$) can be salvaged in special situations. An application would be to the low-energy sulfur ions in Jupiter's Io plasma torus, for which $u_E \gtrsim v_\phi$, where $u_E$ is the component of the corotational velocity perpendicular to the magnetic field line. At Io $u_E$ is 74 km/s, while Io moves at 17 km/s, giving a sulfur ion picked up cold from Io a gyro velocity $v_\phi = 57$ km/s. Several authors [Siscoe, 1977; Göertz and Thomsen, 1979] have assumed the invariance of $J$ during diffusion of plasma injected by Io and other Jovian moons.

In this paper we will study nonrelativistic particles only and derive adiabatic equations of motion as seen by an observer in the corotating reference frame. We will show that $J$ is in fact still conserved even if $u_E \gtrsim v_\phi$, provided the electric and magnetic fields are those of a 'rigid rotator.' A rigid rotator is defined as a time dependent magnetic field configuration, which at any time is the same as that at an earlier time but rotated about an axis through an arbitrary angle (and translated if desired). The electric field is assumed to have the corotational value [Birmingham and Northrop, 1979, equation 28] $E = (r \times \Omega) \times B/c$, where $\Omega$ is the rotational angular velocity of the rigid rotator and $r$ is the vector to the observation point from an origin on the rotating axis (see Figure 1). The rotator may be oblique. Even if the rotating magnetic field pattern should have an intrinsic axis of symmetry (such as a dipole has), this axis need not be aligned with $\Omega$ (see Hones and Bergeson [1965] for an example of this geometry). But an intrinsic symmetry axis need not even exist for what follows below. The usual adiabatic requirements are still needed, i.e., that the gyroradius be small.
Fig. 1. Depiction in cross section of an oblique magnetic rotator. The magnetic field need not have an axis of symmetry. Here \( r \) is the vector displacement from the spin axis \( \Omega \); \( \phi \) is in the azimuthal vector position of a particle with respect to the origin 0 and \( p \) is its by-product of the work will be a clearer understanding of why the magnetic mirror field with \( E = 0 \), where \( J \) is conserved by the changes in trapped particle kinetic energy. Other studies of this fact that \( uE \neq vg \) can be constructed as follows. Take a static magnetic mirror field with \( E = 0 \), where \( J \) is conserved by the drifts. Then look at this arrangement from a reference frame translating with respect to the static frame at a velocity \( \geq v_\phi \). An observer in this translating frame sees \( uE \geq v_\phi \) and yet \( J \) is conserved. Another (trivial) special case where \( J \) is conserved in spite of the drifts, even though he sees no electric field drift. One must show that these two additional drifts do not destroy the invariance of \( J \). The motion as observed in the rotating system is almost periodic, so the invariance of \( J \) can be suspected.

The rigid rotator is not such a trivial special case, because an observer in the corotating frame sees Coriolis and centrifugal drifts, even though he sees no electric field drift. One must show that these two additional drifts do not destroy the invariance of \( J \). The motion as observed in the rotating system is almost periodic, so the invariance of \( J \) can be suspected.

In the following we derive the guiding center equations of motion in a rigid rotator and prove the invariance of \( J \). A by-product of the work will be a clearer understanding of why rigid rotators of arbitrary geometry never produce long-term changes in trapped particle kinetic energy. Other studies of this matter have been made. Hones and Bergeson [1965] proved that a magnetized sphere rotating about an axis not aligned with the dipole produces no secular energy changes. Birmingham and Northrop [1968] extended this result to an arbitrarily moving magnet with the same result. Both papers assumed \( E \cdot B = 0 \), as we will here, but here we will drop the assumption that \( uE \ll v_\phi \) made in both of those papers.

2. PARTICLE MOTION IN THE ROTATING FRAME

The charged particle equation of motion in the nonrotating frame is, in the usual notation

\[
\frac{dv}{dt} = \frac{e}{m} E(r, t) + \frac{e}{mc} v \times B(r, t)
\]

Let \( w = v - \Omega \times r \), so that \( w \) is the particle velocity seen by the observer riding with the rotator. Then \( \frac{dw}{dt} = \frac{dw}{dt} + \frac{d}{dt} (\Omega \times r) \). Let the rate of change of a vector as seen by the corotating observer be denoted by \( (d/dt)c \). Then [Goldstein, 1950, equation 4.105],

\[
(d/dt)c = (d/dt) - \Omega \times \frac{dw}{dt}
\]

Because \( \Omega \times (\Omega \times r) = -\frac{1}{2} r^2 \Omega^2 \), where \( r \) is the distance (Figure 1) from the rotation axis to the particle position, \( (d/dt)c \) becomes

\[
\frac{dw}{dt} = -2 \Omega \times w + \frac{1}{2} r^2 \Omega^2 + \frac{eE}{m} (r, t) + \frac{e}{mc} (w + \Omega \times r) \times B(r, t)
\]

E still being as observed from the nonrotating frame. Equation (3) is quite general, requiring only that \( \Omega \) be time and space independent. For a rigid rotator [Birmingham and Northrop, 1979],

\[
E = (r \times \Omega) \times B/c = - (\rho \Omega/\phi) \times B
\]

where \( \phi \) is the unit azimuthal vector about the \( \Omega \) axis. For this electric field, (3) becomes

\[
\frac{dw}{dt} = \frac{1}{2} \rho^2 \Omega^2 + \frac{e}{mc} w \times \left[ B(r) + \frac{2mc}{e} \Omega \right]
\]

In the rotating frame the electric field vanishes, and \( B \) is time independent as indicated. Equation (4) contains both centrifugal and Coriolis forces. Dotting (4) with \( w \) gives

\[
\frac{1}{2} \frac{d}{dt} (w^2 - \rho^2 \Omega^2) = 0
\]

so that \( H \equiv \frac{1}{2} (w^2 - \rho^2 \Omega^2) = \frac{1}{2} v^2 - \rho \Omega \cdot \phi \) is an exact constant of the particle motion. Here \( \frac{1}{2} \rho^2 \Omega^2 \) is the centrifugal potential, and the particle energy seen by the rotating observer is larger when the particle finds itself further from the rotation axis. But (5) does not place any constraint on how far from the rotation axis the particle can find itself; hence it places no constraint on energy. So far this is all well known. What is new is that the second invariant \( J \) still holds and places a constraint on where the particle goes, hence on \( \rho^2 \Omega^2 \) and consequently on \( w^2 \) and \( v^2 \).

Since \( (dw/dt)c = w \), equation (4) is formally identical to the equation of motion of a particle in a static electric potential \(-m/2e)\rho^2 \Omega^2 \) and a static magnetic field \( E = B + (2mc/e)\Omega \). The electric field in this analogy is \( \rho \Omega/\phi \), and \( \phi \) satisfies the Maxwell equations \( \nabla \times \rho \Omega/\phi = - (1/c) \partial B/\partial t = 0 \) and \( \nabla \cdot \rho \Omega/\phi = 0 \), both of which are repeatedly used in adiabatic theory.

If there exists an electric field \( \delta E \) in addition to that which produces corotation, it can be added to the right side of equation (4). Of course, \( H \) is modified: if \( \delta E \) is derivable from a potential \( \delta E = -\nabla \phi \), \( \delta E/m \) must be added to maintain its constancy; if \( \nabla \times \delta E \neq 0 \), there is no longer an 'energy constant' \( H \).

The formal analogy between (1) and (4) permits application of all heretofore developed adiabatic theory to particle dynamics in the rotating frame, as described by (4). Because \( \delta E \) is proportional to the adiabatic parameter \( \epsilon = m/c \), only the small electric field form of adiabatic theory is required, and this fact...
provides much simplification compared to working from the point of view of the non-rotating frame with its large E and attendant more complex formalism.

By making use of (4), the Vlasov equation can be derived for the rotating frame. Let \( f(r, w, t) \) be the particle distribution function in \((w, r)\) space. By conservation of particles

\[
\frac{\partial f}{\partial t} + V_r \cdot (f w) + V_w \cdot f \left( \frac{dw}{dt} \right) = 0
\]  

(6)

From (4), \( V_r \cdot (dw/dt) = 0 \). Furthermore, \( V_r \cdot w = 0 \). Thus (6) becomes a Vlasov equation in the rotating world:

\[
\frac{\partial f}{\partial t} + w \cdot V f + \frac{1}{2} \rho^2 \Omega^2 + \frac{e w}{mc} \times \mathbf{B} \cdot V \mathbf{f} = 0
\]  

(7)

Application of the analogy to the usual Vlasov equation also yields (7) directly.

Anisotropies of the low energy particles in rigid rotators may be analyzed in the rotating system by the techniques of Birmingham and Northrop [1979] and of Northrop and Thomsen [1980] and then converted to any other frame of reference such as the spacecraft frame [see Northrop, 1976]. This procedure circumvents carrying out adiabatic solutions of the Vlasov equation in the nonrotating spacecraft frame, where \( u_g \geq v_p \). If \( u_g \geq v_p \), many terms that were dropped by the above authors as small would have to be retained, thereby greatly expanding an already extensive analysis.

In the remainder of this paper we use the analogy repeatedly to obtain equations valid for the corotating observer.

3. THE GUIDING CENTER EQUATION OF MOTION

In a nonrotating frame the differential equation of motion of the guiding center located at \( \mathbf{R} \) is [Northrop, 1963, equation 1.12]

\[
\dot{\mathbf{R}} = \frac{e}{m} \mathbf{E} \mathbf{(R, t)} + \frac{\epsilon}{mc} \mathbf{R} \times \mathbf{B} \mathbf{(R, t)} - \frac{M}{m} \mathbf{V} \mathbf{B} \mathbf{(R, t)} + \theta(\epsilon)
\]  

(8)

where \( \epsilon = m/e \) and \( \theta(\epsilon) \) means terms of that order which have not been written down; superposed dots stand for total time derivatives. The analog of (8) is

\[
\dot{\mathbf{R}}_c = \frac{e}{m} \mathbf{e} + \frac{\epsilon}{mc} \mathbf{R}_c \times \mathbf{B} - \frac{M}{m} \mathbf{V} \mathbf{B} + \theta(\epsilon)
\]  

(9)

where \( \mathbf{R}_c = (d\mathbf{R}/dt)_c \) and \( \dot{\mathbf{R}}_c = (d^2\mathbf{R}/dt^2)_c \). Conversion of (9) to the actual fields is made by the definitions of \( \mathbf{e} \) and \( \mathbf{B} \).

\[
\dot{\mathbf{R}}_c = \nabla_r \rho^2 \Omega^2 + \frac{\epsilon}{mc} \mathbf{R}_c \times \mathbf{B} + \frac{2M}{m} \mathbf{V} \mathbf{B} + \theta(\epsilon)
\]  

(10)

\( M \), the magnetic moment, is a scalar property of the particle and not frame dependent.

4. THE DRIFT VELOCITY

The drift velocity as seen by a corotating observer may be derived in the following three ways (in increasing order of difficulty): (1) by solving (10) for \( \mathbf{R}_{c1} \), the component of \( \mathbf{R}_c \) perpendicular to \( \mathbf{B} \), by crossing (10) with \( \mathbf{B} \); (2) by using the usual drift velocity valid for \( u_g \leq v_p \) plus for analogy; (3) by starting with the general eight-term drift expression [Northrop, 1963, equation 1.17] valid when \( u_g \leq v_p \), and substituting into it \( \mathbf{E}(r, t) \) and \( \mathbf{B}(r, t) \) for the rigid rotator. We have carried out all three and give the second one here to again illustrate use of the analogy. The guiding center velocity in an inertial system is (when \( u_g \ll v_p \)) the familiar expression

\[
\dot{\mathbf{R}}_c = \frac{\epsilon}{mc} \mathbf{e} + \frac{\epsilon}{mc} \mathbf{e} \left( \frac{1}{2} \rho^2 \Omega^2 + \frac{M}{m} \mathbf{V} \mathbf{B} \right)
\]  

(11)

where \( \dot{\mathbf{R}}_c = \mathbf{B} / \mathbf{B} \) and \( v_p = \dot{\mathbf{R}}_c \cdot \mathbf{B} \). Let \( s \) be distance along a \( \mathbf{B} \) line. Then \( \dot{\mathbf{R}}_c \cdot \mathbf{B} = \partial \dot{\mathbf{R}}_c / \partial s \). We caution that (11) is not itself valid from the point of view of the non-rotating observer in our present problem, because it assumes that \( u_g \ll v_p \), whereas just the reverse is true for him. But (11) is the expression whose analog will lead to the rotating frame equation (15) and to equation (17) which is valid for the nonrotating observer. We can draw the analogy from an \( u_g \ll v_p \) equation because the analog electric field \( \mathbf{e} \) in the rotating system leads to a centrifugal drift velocity (see equation (16)) which is much less than the particle velocity \( w \) so long as \( \Omega \ll \omega \).

By the analogy, the guiding center velocity \( \dot{\mathbf{R}}_c \) seen by an observer in the corotating frame is

\[
\dot{\mathbf{R}}_c = \frac{\epsilon}{mc} \mathbf{e} + \frac{\epsilon}{mc} \mathbf{e} \left( \frac{1}{2} \rho^2 \Omega^2 + \frac{M}{m} \mathbf{V} \mathbf{B} \right)
\]  

(12)

where \( \omega = eB/mc \sim \epsilon^{-1} \). When (13) is expanded through first order in \( \epsilon \), one finds

\[
\dot{\mathbf{R}}_c = \frac{\epsilon}{mc} \mathbf{e} + \frac{\epsilon}{mc} \mathbf{e} \left( \frac{1}{2} \rho^2 \Omega^2 + \frac{M}{m} \mathbf{V} \mathbf{B} \right)
\]  

(14)

where \( u_{c1} \equiv \dot{\mathbf{R}}_c \cdot \mathbf{B} \). We have used the fact that \( (2/\omega)\Omega_\perp \cdot \dot{\mathbf{R}}_c = (2/\omega)\Omega_\perp \cdot \dot{\mathbf{R}}_c \cdot \mathbf{B} = \theta(\epsilon^2) \), because as will be seen below in (16), \( \dot{\mathbf{R}}_{c1} \) is distance along lines of the analog field \( \mathbf{B} \). Through first order in gyroradius, the last term of (12) may just as correctly be written with the substitution of \( \dot{\mathbf{R}}_c \mathbf{B} \) for \( \mathbf{B} \) because \( \mathbf{B} = \mathbf{B} + (2mc/e)\mathbf{B} = \mathbf{B} + \theta(\epsilon) \), and \( \mathbf{B} = \mathbf{B} / \mathbf{B} = \dot{\mathbf{R}}_c + 2\Omega_\perp / \omega = \dot{\mathbf{R}}_c + 2\Omega_\perp / \omega = \dot{\mathbf{R}}_c + \theta(\epsilon^2) \) (where \( \Omega_\perp = \dot{\mathbf{R}}_c \times \mathbf{B} \)). The error made is of \( \theta(\epsilon^2) \), because the expression is brackets is already explicitly of \( \theta(\epsilon^2) \), since \( M/e \) and \( \epsilon \) are both proportional to \( m/e \). The guiding center velocity in the rotating frame then is (with \( \mathbf{e} = m/e \nabla_r \rho^2 \Omega^2 \))

\[
\dot{\mathbf{R}}_c = \frac{\epsilon}{mc} \mathbf{e} + \frac{\epsilon}{mc} \mathbf{e} \left( \frac{1}{2} \rho^2 \Omega^2 + \frac{M}{m} \mathbf{V} \mathbf{B} \right)
\]  

(15)

where \( \dot{\mathbf{R}}_{c1} = \dot{\mathbf{R}}_c / \omega \). The guiding center velocity in the rotating frame then is (with \( \mathbf{e} = m/e \nabla_r \rho^2 \Omega^2 \))

\[
\dot{\mathbf{R}}_c = \frac{\epsilon}{mc} \mathbf{e} + \frac{\epsilon}{mc} \mathbf{e} \left( \frac{1}{2} \rho^2 \Omega^2 + \frac{M}{m} \mathbf{V} \mathbf{B} \right)
\]  

(16)
The drift velocity (16) shows the expected gradient and line curvature drifts, plus a centrifugal and Coriolis drift, the latter being proportional to the first power of the parallel velocity. The guiding center velocity \( \dot{\mathbf{R}} \) as seen by a nonrotating observer is \( \dot{\mathbf{R}} \), plus the corotational velocity \( \mathbf{\Omega} \times \mathbf{R} = \rho \mathbf{O} \phi \).

\[
\dot{\mathbf{R}} = \dot{e}_i u_{ii} + \dot{\mathbf{R}}_{\perp} + \rho \mathbf{O} \phi \\
= \dot{e}_i (u_{ii} + \rho \mathbf{O} \phi \cdot \dot{\mathbf{e}}_i) + \rho \mathbf{O} \phi (\dot{\mathbf{e}}_i \cdot \dot{\mathbf{e}}_i) + \dot{\mathbf{R}}_{\perp}
\]

(17)

The last form because \( \mathbf{E} = -\rho \mathbf{O} \phi \times \mathbf{B} / c \). The parallel velocity seen by the nonrotating observer is \( u_{\parallel} = \dot{e}_i \cdot \dot{\mathbf{R}} = u_{ii} + (\mathbf{\Omega} \times \mathbf{R}) \cdot \dot{\mathbf{e}}_i = u_{ii} + \rho \mathbf{O} \phi \cdot \dot{\mathbf{e}}_i \).

In deriving (16) we have gone through the following routine: (1) written the desired adiabatic equation for a nonrotating system for the case where \( \epsilon \ll \epsilon_0 \); (2) written down the analog by the prescription \( \mathbf{B} \rightarrow \mathbf{a} \), \( \mathbf{E} \rightarrow \mathbf{e} \), \( \dot{\mathbf{e}}_i \rightarrow \dot{\mathbf{e}}_i \), \( s \rightarrow \sigma \), \( u_{ii} \rightarrow u_{ii} \), \( \dot{\mathbf{R}} \rightarrow \dot{\mathbf{R}}_e \); (3) converted the analog equation to actual quantities seen by the rotating observer by means of \( \dot{\mathbf{b}} = \dot{\mathbf{e}}_i + 2 \mathbf{O} \sigma / \epsilon + (\epsilon_0) \), \( \mathbf{B} = \mathbf{B} + 2 \mathbf{mc} / \epsilon \mathbf{O} \sigma \), \( \mathbf{E} = \mathbf{E} + \mathbf{O} \sigma (\mathbf{V} \mathbf{R}^2) / \epsilon \), all of which are easily derived; and (4) finally, if desired, converted the rotating frame equation to an equation analogous in the nonrotating frame by \( \dot{\mathbf{R}} = \dot{\mathbf{R}}_e + \rho \mathbf{O} \phi \), and \( \dot{\mathbf{R}} = \dot{\mathbf{R}}_e \). This latter comes from using twice the relationship between total time derivatives of vectors, \( d/dt = (d/dt)_e + \mathbf{O} \times \).

The four steps (1)-(4) are illustrated by equations (11), (12), (15), and (17), respectively. It is important not to confuse an expression like (11), from which the analog is drawn, with (17), which applies to the real situation seen by the nonrotating observer. It is also important to distinguish among \( u_{ii} \) (guiding center velocity parallel to \( \mathbf{B} \) seen in the rotating system), \( v_{ii} \) (guiding center velocity parallel to \( \mathbf{B} \) seen from the nonrotating system), and \( \dot{\mathbf{e}}_i \cdot \dot{\mathbf{R}}_e \) (guiding center velocity in the rotating system parallel to \( \mathbf{a} \), a 'parallel velocity' for which we have not introduced a symbol). Possible confusion may also arise between \( u_{ii} \) and \( ds/dt \) or between \( \dot{\mathbf{b}} \cdot \dot{\mathbf{R}}_e \) and \( ds/dt \). The surfaces of constant \( s \) or \( \sigma \) are not necessarily perpendicular to \( \mathbf{B} \) or \( \mathbf{a} \), respectively. The necessary and sufficient condition that a set of surfaces normal to a divergence-free field such as the magnetic field even exist is that \( \mathbf{B} \cdot \mathbf{V} \times \mathbf{B} = 0 \) [Newcomb, 1958, 1980]. That is, there can be no current parallel to \( \mathbf{B} \). And even if that condition should be fulfilled, the orthogonal surfaces are not constant \( s \) surfaces, except in highly symmetric geometry, to which we do not wish to confine ourselves. Thus in deriving (18) before taking its rotating frame analog. The rate of change of the parallel speed is \( d/dt (\dot{e}_i \cdot \dot{\mathbf{R}}) \), and this differs from (18) by

\[
\dot{\mathbf{R}} \cdot d\mathbf{e}_i/dt = (\dot{e}_i \cdot \dot{\mathbf{R}}) \cdot d\mathbf{e}_i/dt = \dot{\mathbf{R}}_{\perp} \cdot d\mathbf{e}_i/dt = \dot{e}_i \cdot \dot{\mathbf{R}}_e \cdot d\mathbf{e}_i/d\sigma + \theta (\epsilon^2)
\]

where \( \dot{\mathbf{R}}_e \) is the usual three term \((\mathbf{E} \times \mathbf{B}, \text{gradient, and curvature})\) drift velocity. One finds that

\[
\dot{\mathbf{R}} \cdot d\mathbf{e}_i/dt = -(\dot{e}_i \cdot \dot{\mathbf{R}}_e / (\epsilon_0) \cdot \dot{\mathbf{e}}_i \cdot \dot{\mathbf{e}}_i / (\epsilon_0)) \cdot \left( \mathbf{V} \mathbf{B} (m - e\mathbf{E}/m) + \theta (\epsilon^2) \right)
\]

and with the use of (18),

\[
\frac{d}{dt} (\dot{e}_i \cdot \dot{\mathbf{R}}_e) = \frac{\mathbf{M}_0 + e\mathbf{M} \cdot \mathbf{G} \cdot \mathbf{E}}{m} \cdot \frac{\mathbf{G} \cdot \mathbf{E}}{m} + \frac{e}{m} \cdot \mathbf{E} \cdot \mathbf{E}
\]

(19)

Equation (20) and its drift velocity companion (11) are the two equations that determine guiding center motion in ordinary, small \( \epsilon \), inertial reference frame situations. We again caution, just as after (11), that (20) is not itself valid from the point of view of the nonrotating observer in our present problem but is the expression whose analog leads to the rotating frame equation (27) that we want. Equation (30) is the one valid from the point of view of the nonrotating observer. The rotating frame analog of (20) is

\[
\frac{d}{dt} (\dot{e}_i \cdot \dot{\mathbf{R}}_e) = \frac{\mathbf{M}_0 + e\mathbf{M} \cdot \mathbf{G} \cdot \mathbf{E}}{m} \cdot \frac{\mathbf{G} \cdot \mathbf{E}}{m} + \frac{e}{m} \cdot \mathbf{E} \cdot \mathbf{E}
\]

(21)

In (21) and in other equations below, \( d/dt \) does not carry the subscript \( c \) because the time derivative of a scalar is the same in either reference frame, so the subscript is not needed. \( \mathbf{G} = \mathbf{B} + (2mc/e) \mathbf{O} \sigma \) and \( \mathbf{G} \mathbf{E}/m = \mathbf{V} \mathbf{B} (2p^2) \) are now substituted into (21) to get the parallel equation of motion along the \( \mathbf{B} \) lines in the rotating frame. The following are needed and can easily be verified:

\[
\dot{\mathbf{b}} = \dot{\mathbf{e}}_i + \frac{2\Omega \mathbf{b}}{\omega} + \theta (\epsilon^2)
\]

(22)

\[
\mathbf{G} = \mathbf{B} \left( 1 + \frac{2\dot{\mathbf{e}}_i \cdot \mathbf{O} \sigma}{\omega} \right) + \theta (\epsilon^2)
\]

(23)

\[
\dot{\mathbf{b}} \cdot \mathbf{V} \mathbf{G} = \frac{\partial \mathbf{G} \cdot \mathbf{B}}{\partial \sigma} \cdot \frac{2\Omega \mathbf{b}}{\omega} + \left( \mathbf{V} \mathbf{B} + \mathbf{B} \dot{\mathbf{e}}_i / \partial \sigma \right) + \theta (\epsilon^2)
\]

(24)

\[
\dot{\mathbf{b}} \cdot \mathbf{V} \mathbf{b} \mathbf{b} = \rho \mathbf{b} \mathbf{b} \cdot \mathbf{V} \mathbf{b} + \Omega \mathbf{b} \mathbf{b} \cdot \mathbf{p}
\]

(25)

where \( \rho \) is as in Figure 1. The left side of (21) must also be converted:

\[
\frac{d}{dt} (\dot{e}_i \cdot \dot{\mathbf{R}}_e) = \frac{d}{dt} (\dot{e}_i \cdot \dot{\mathbf{R}}_e) + \theta (\epsilon^2)
\]

(26)
the last equality again arising from the fact that
\[
(2/o)\Omega_1 \cdot \dot{R}_e = (2/o)\Omega_1 \cdot \dot{e}_1 = \omega (c^2)
\]
The last term in (21) is of order \( e \) because of \( m/e \) in its coefficient and may therefore have \( \hat{\theta}, \hat{\Phi}, \) and \( \sigma \) replaced by \( \hat{\epsilon}_1, \hat{B}, \) and \( \phi, \) respectively. Then
\[
d\dot{u}_1 = -\frac{M}{m} \frac{\partial B}{\partial s} + \frac{\dot{B}}{m} + \frac{\partial}{\partial s} \frac{1}{2} \rho^2 \Omega^2
\]
\[
-\frac{1}{2} \left( u_1 \dot{e}_1 \times \frac{\partial e_1}{\partial s} + 2 \Omega_1 \right) \cdot \nabla \left( \frac{MB}{m} - \frac{1}{2} \rho^2 \Omega^2 \right)
\]
\[
- \frac{2}{m} \frac{MB}{\Omega_1} \frac{\partial e_1}{\partial s} + \theta (c^2)
\]
(27)
where \( M = M_0 + \epsilon M_1, \) the right-hand side of (27) contains the expected mirror and centrifugal forces, as well as two terms that are first order in gyroradius.

Through lowest order, (27) has the energy integral for the guiding center velocity along \( \hat{B} \)
\[
K_c = \frac{m}{2} (u_{12}^2 - \rho^2 \Omega^2) + MB = \text{const}
\]
consistent with the exact integral (5), \( MB \) being the gyration energy. Or more explicitly, \( K_c / m = 0 + \theta (c^2). \)

The parallel equation of motion actually valid from the viewpoint of the nonrotating observer, who sees \( \hat{u}_B \geq v_\parallel \) may be obtained from (27) by the addition of \( d/dt(\rho \Omega_1 \cdot \dot{e}_1) \) to it, because \( v_\parallel = u_1 + \rho \Omega_1 \cdot \dot{e}_1 \). We give the final expressions only and do not include the proofs.

\[
d\dot{u}_1 = -\frac{M}{m} \frac{\partial B}{\partial s} + \frac{\dot{B}}{m} + \frac{\partial}{\partial s} \frac{1}{2} \rho^2 \Omega^2
\]
\[
-\frac{1}{2} \left( u_1 \dot{e}_1 \times \frac{\partial e_1}{\partial s} + 2 \Omega_1 \right) \cdot \nabla \left( \frac{MB}{m} - \frac{1}{2} \rho^2 \Omega^2 \right)
\]
\[
- \frac{2}{m} \frac{MB}{\Omega_1} \frac{\partial e_1}{\partial s} + \theta (c^2)
\]
(27)

6. THE LONGITUDINAL INVARIANT

In an inertial system, when \( u_e < v_\parallel \) the second or longitudinal invariant is defined as
\[
J = \int \frac{f}{\rho} ds - m \dot{u}_1 \frac{v_\parallel}{ds} ds
\]
(31)
If for static \( E \) and \( B \) fields, \( K \) is defined as \( (m v_\parallel^2)/2 + MB + e\phi \), where \( \phi \) is the electrostatic potential, then by (20), \( K/m = 0 + \theta (e^2) \), so that in 'ordinary' (i.e., nonrotating reference frame and \( u_e < v_\parallel \) adiabatic theory, \( J \) can be equally well defined for static fields by
\[
J = \int \left[ \frac{1}{m} (K - MB - e\phi) \right]^{\frac{1}{2}} ds
\]
(32)
The analogs in the rotating frame of (31) and (32) are
\[
J_r = \int \left[ \frac{1}{m} (K - MB - e\phi) \right]^{\frac{1}{2}} ds
\]
(33)
where \( \mathcal{X}/m = \frac{1}{2} (b \cdot \dot{R}_e)^{-1} + \mathcal{F} \) being the bounce period \( \mathcal{F} = \frac{d}{d\phi} \mathcal{F}^1 (b \cdot \dot{R}_e)^{-1} \) is conserved over drift time scales.

Suppose instead one would like to use as invariant in the rotating system the more conventional integral along the actual magnetic field \( \hat{B} \). The question is, does the quantity
\[
J_c = m \dot{u}_1 \frac{v_\parallel}{ds} \cdot \dot{R}_e = m \dot{u}_1 \frac{ds}{u_1}
\]
(35)
satisfy \( \langle d/dt \rangle J_c/m = 0 + \theta (c^2) \)? In (35), the integral is now along \( B \) lines and angle brackets mean \( T_c \cdot \frac{d}{d\phi} \mathcal{F}^1 (b \cdot \dot{R}_e)^{-1} \). This is conserved over drift time scales.

A crucial point turns out to be that the Coriolis drift is proportional to \( \dot{\Omega}_1 \), which reverses sign between the two directions of integration along the field line in performing the round trip integration \( \int ds \ldots \). That is, an integral of the form \( \int ds u_1 n \cdot h(s) \) vanishes if \( n \) is odd (Coriolis drift) but does not vanish for \( n \) even (line curvature and gradient drifts). \( J_c/m \) may be written as
\[
J_c(\alpha, \beta, \mathcal{K}_c, M) = \int ds \cdot \left[ \frac{2}{m} \left( \mathcal{K} - MB(\alpha, \beta, \mathcal{S}) \right) + \Omega^2 \rho^2 (\alpha, \beta, \mathcal{S}) \right]^{\frac{1}{2}}
\]
(36)
where $K_r$ is defined in (28), and where $M = M_0 + \varepsilon M_1$. Here $\alpha(r, t)$ and $\beta(r, t)$ are the usual Eulerians [Northrop and Teller, 1960; Stern, 1970] constant on a field line and such that $\mathbf{B} = \mathbf{V} \times \nabla \beta$, with the vector potential $\Lambda = \alpha \mathbf{V}$. Even for a given field, $\alpha$ and $\beta$ are nonunique. Of the possible choices for the rigid rotator, we select them so that the $\alpha$, $\beta$ labels of a given rotating field line do not change with time. Then time does not appear explicitly in the argument of $B$ in (36). The rate of change of $J_c/m$ is

$$\frac{d J_c}{dt} = \frac{1}{m} \left[ K_r \frac{\partial J_c}{\partial r} + \frac{\partial J_c}{\partial \alpha} + \beta \frac{\partial J_c}{\partial \beta} + M \frac{\partial J_c}{\partial M} + \frac{\partial J_c}{\partial t} \right]$$

(37)

Now $\partial J_c/\partial t = 0$ and $M \equiv d/(dt)(M_0 + \varepsilon M_1) = \varepsilon(\varepsilon^2)$, and $\alpha$ and $\beta$, being constant on a field line, are changed only by the drift velocity. By an analysis quite similar to that given by Northrop [1963], an analysis which will not be repeated here, one can derive the drift velocity (16) causes $\alpha$ and $\beta$ to change at rates given by

$$e \frac{d \alpha}{dt} = -e \frac{d}{dt} \left( u_\parallel \frac{\partial R}{\partial \beta} \right)$$

$$-\frac{\partial}{\partial \beta} \left( \frac{MB}{m} - \frac{1}{2} p^2 \Omega^2 \right) + 2u_\parallel \frac{\Omega \cdot \mathbf{V} x}{B} + \varepsilon(e^2)$$

$$e \frac{d \beta}{dt} = e \frac{d}{dt} \left( u_\parallel \frac{\partial R}{\partial \alpha} \right)$$

$$+ \frac{\partial}{\partial \alpha} \left( \frac{MB}{m} - \frac{1}{2} p^2 \Omega^2 \right) + 2u_\parallel \frac{\Omega \cdot \mathbf{V} x}{B} + \varepsilon(e^2)$$

(38)

where $R(\alpha, \beta, \gamma)$ is the guiding center position. The bounce averages are

$$\langle \dot{\alpha} \rangle = \frac{1}{T} \int_{u_\parallel} \text{ds} \dot{\alpha} = -\frac{mc}{eT} \int_{u_\parallel} \text{ds} \frac{\partial}{\partial \beta} \left( \frac{MB}{m} - \frac{1}{2} p^2 \Omega^2 \right) + \varepsilon(e^2)$$

$$\langle \dot{\beta} \rangle = \frac{1}{T} \int_{u_\parallel} \text{ds} \dot{\beta} = \frac{mc}{eT} \int_{u_\parallel} \text{ds} \frac{\partial}{\partial \alpha} \left( \frac{MB}{m} - \frac{1}{2} p^2 \Omega^2 \right) + \varepsilon(e^2)$$

(39)

where $T$ is the bounce period along $B$.

By comparison with (36),

$$\langle \dot{\alpha} \rangle = \frac{c}{eT} \frac{\partial J_c}{\partial \beta} + \varepsilon(e^2)$$

$$\langle \dot{\beta} \rangle = -\frac{c}{eT} \frac{\partial J_c}{\partial \alpha} + \varepsilon(e^2)$$

(40)

In deriving (39) by bounce averaging (38), the integral

$$\frac{1}{T} \int_{u_\parallel} \left( \frac{d}{dt}(u_\parallel \frac{\partial R}{\partial \beta}) \right)$$

vanished because $d\text{ds}/dt = dt$ to lowest order in $e$, and $u_\parallel$ vanishes at the mirror points. Furthermore, the Coriolis term (last term in (38)) bounce averaged to zero, being an example of an odd $n$ integral.

Finally, we must examine $K_r$ and its bounce average. From the definition (28) of $K_r$

$$\frac{1}{m} \frac{dK_r}{dt} = u_\parallel \frac{du_\parallel}{dt} + M \frac{dB}{dt} - \frac{d}{dt} \left( \frac{1}{2} p^2 \Omega^2 \right)$$

(41)

where $dB/dt = (\dot{\alpha}, u_\parallel + \dot{R}_c) \cdot \mathbf{V}$ and similarly for $(d/dt) \frac{1}{2} p^2 \Omega^2$. $\dot{R}_c$ is given by (16) and $du_\parallel/dt$ by (27). With these substitutions into (41) and straightforward vector algebra, one finds

$$\frac{1}{m} \frac{dK_r}{dt} = \frac{2}{e} \frac{\varepsilon}{u_\parallel} \frac{\partial}{\partial \Omega} \frac{\partial}{\partial \alpha} + \varepsilon(e^2)$$

(42)

Consequently, $\langle K_r/m \rangle = 0 + \varepsilon(e^2)$ because $K_r$ is proportional to an odd power of $u_\parallel$. Thus with some modification of the usual direct proof of the invariance of $J$ in a nonrotating, small $u_\parallel$ situation, we have shown that the bounce average of (37) vanishes here also in the rapidly rotating frame:

$$\langle \frac{d J_c}{dt} \rangle = 0 + \varepsilon(e^2)$$

(43)

Therefore, both $\mathcal{J}$ and $J_c$ are conserved by the first order in gyroradius guiding center motion. The invariance of $J_c$ prescribes a shell on which the guiding center slowly drifts in the fashion so familiar for static, small $u_\parallel$, nonrotating systems. The only difference is that the drift shell is now rapidly rotating about $\Omega$.

When (36) is used to find a drift shell, $K_r$ should be held constant. This is because by (42) $\langle K_r \rangle = 0$ on the drift time scale. The rigid rotator is a fortunate special case in this respect, the fields being static in the corotating frame, even though time dependent to the nonrotating observer. Ordinarily when fields are time dependent, the appropriate $K$ is not a constant as the particle drifts, even though the time dependence and $u_\parallel$ are small enough to conserve the second invariant $d$. In such cases $K$ must be followed in time by integrating the relationship [see Northrop, 1963, equation 3.62] $\langle K_r \rangle = -\frac{1}{T} \int_0^T \frac{d}{dt}(\frac{1}{T} \int_0^T \frac{d}{dt})$.

The right-hand side of (42) equals $-\frac{2}{e} \frac{\varepsilon}{u_\parallel} \frac{\partial}{\partial \Omega} \frac{\partial}{\partial \alpha} + \varepsilon(e^2)$ because $u_\parallel \partial/\partial t = dt/\partial \Omega + \varepsilon(e^2)$, the time derivative being that seen by an observer moving with the guiding center. Thus $m \frac{1}{2} \frac{d}{dt}(K_r + 2Mc/e) \frac{\partial}{\partial \Omega} \frac{\partial}{\partial \alpha} = 0 + \varepsilon(e^2)$, and $\frac{1}{2} u_\parallel^2 - \frac{1}{2} p^2 \Omega^2 + MB/m + 2(Mc/e) \Omega \cdot \dot{\Omega} \cdot \cos$ is conserved by the total guiding center motion $\dot{R}_c$ (parallel velocity plus drifts), whereas $K_r$ is constant only if the drifts are ignored. In finding a drift shell from (36), $K_r + 2(Mc/e) \Omega \cdot \dot{\Omega}$ may be used in place of $K_r$, and is in fact $\mathcal{J}$.

7. PARTICLE ENERGIZATION

Previous direct calculations [Hones and Bergeson, 1965; Birmingham and Northrop, 1968] of particle energization in rigid rotators have never revealed a systematic energy change. The Hones-Bergeson paper investigated the special case of a dipole, field rotating about an $\Omega$ axis oblique to but intersecting the dipole axis. The Birmingham-Northrop paper generalized that paper to arbitrary fields moving in an arbitrary fashion. Both papers assumed that the motion was so slow that $ue < \varepsilon$. In the present work we have not made the restrictive assumption that the electric field is small. Let the exact constant of the motion $\frac{1}{2}(w^2 - \rho^2 \Omega^2)$ be denoted by $H$. Gyroaveraging gives

$$\left\{ \frac{w^2}{2} \right\} = H + \frac{\rho^2 \Omega^2}{2} + \varepsilon(e^2)$$

(44)

where as a consequence of the gyroaveraging, $\rho$ is now at the guiding center position $R$ rather than at the particle position. The bounce average of (44) is $H = \mathcal{J} + \varepsilon(e^2)$.

$$\left\{ \frac{w^2}{2} \right\} = H + \frac{\rho^2 \Omega^2}{2} + \varepsilon(e^2)$$

(45)

From the point of view of the corotating observer, the particle's kinetic energy changes periodically in accordance with (44)
as the particle moves rapidly back and forth along field lines, thereby changing $p^2\Omega^2/2$ periodically. Superposed on this periodicity is a longer term change caused by the slower drift around the drift shell, the shell being defined by the invariance of $J_c$. The bounce averaged energy (45) changes on the drift time scale, in response to changing $\langle p^2\Omega^2/2 \rangle$. However, after one drift period, the guiding center returns to its original $\langle p^2\Omega^2/2 \rangle$, hence to its original $\{(v^2)/2\}$. Thus there can be no long term change in particle energy over many periods of $\Omega$ or over many drift periods around the rapidly rotating drift shell. This conclusion holds even for completely asymmetric rigid rotators. Hones and Bergeson [1965] could have had their centrifugal term $p^2\Omega^2/2$ in (36), hence to its original $(v^2)/2$. Thus there can be no long-term energy changes. So an off-center dipole, such as Jupiter’s, causes no long-term energization, at least insofar as Jupiter meets the criteria of a rigid rotator.

From the point of view of the nonrotating observer or spacecraft, the energy picture is a little different, but the conclusion that no long-term energy changes occur remains valid. The velocity seen by the nonrotating observer is

$$v = w + \Omega \times r = w + p\Omega \phi$$

(46)

Squaring and gyroaveraging leads to

$$\left\{ \frac{v^2}{2} \right\} = \int \left\{ \frac{w^2}{2} \right\} + p^2\Omega^2 + p\Omega \phi \cdot \dot{R}_t + \theta(e^2)$$

with $\{w^2/2\}$ from (44), (47), becomes

$$\left\{ \frac{v^2}{2} \right\} = H + p^2\Omega^2 + p\Omega \phi \cdot \dot{R}_t + \theta(e^2)$$

(48)

and $\phi$ in (47) and (48) being at the guiding center position $\mathbf{R}$, and $\dot{R}_t = \dot{\phi} \cdot u_\parallel + \dot{R}_{\perp}$, as in (15). If the drift $\dot{R}_{\perp}$ is ignored in (48) and just the rapid motion along the field line studied, one has

$$\left\{ \frac{v^2}{2} \right\} = H + p^2\Omega^2 + u_\parallel \cdot p\Omega \phi \cdot \dot{\phi} + \theta(e)$$

(49)

Figure 2 shows schematically how the energy changes occurring during the rapid oscillation along the field line would look for the case of a dipole rotating about an axis perpendicular to the dipole (i.e., $90^\circ$ dipole tilt). Figure 2 applies to a field line lying in the plane perpendicular to $\Omega$. Time is taken as zero at the northern mirror point. The energy gains and losses illustrated by the $(v^2/2) - H$ curve in Figure 2 are caused by Fermi acceleration (types A and B, see Northrop [1963], equation 1.36) contained in $u_\parallel \cdot p\Omega \phi \cdot \dot{\phi}$, and by the centrifugal effect $p^2\Omega^2$. The Fermi acceleration term agrees with Hones and Bergeson [1965], who do not however have the centrifugal term because of their assumption of small $E$. For a field line in the plane defined by $\Omega$ and the dipole, Figure 2 is simpler because $\phi \cdot \dot{\phi} = 0$ everywhere on it and there is no Fermi acceleration, only the centrifugal effect which itself vanishes at the equator. As time progresses, the guiding center drifts around its 'endwise' rotating drift shell. The shell will not be axisymmetric about the dipole because of the $p^2\Omega^2$ term in (36). Here $p^2\Omega^2$ depends not only on dipole latitude but also on the field line. In Figure 2, $p^2\Omega^2$ is smaller at a given dipole latitude for field lines lying in the plane defined by $\Omega$ and the dipole than for those in the plane of the paper. This leads to an asymmetric drift surface in the axisymmetric dipole field.

If the tilt is less than $90^\circ$, e.g., $10^\circ$ as at Jupiter, the energy changes are qualitatively as in Figure 2. Again there is no Fermi acceleration on field lines lying in the plane of $\Omega$ and the dipole because $\phi \cdot \dot{\phi} = 0$ on these lines. A combination of tilt and field line sweepback out of a plane produces more complicated patterns than in Figure 2.

8. The Third Invariant

The third or flux invariant $\Phi$ is conserved here, since the drift shells are static in shape as observed from the rotating frame and so the magnetic flux through them is trivially constant. The third invariant offers no useful information here.

If the rotating magnetic field pattern is not quite rigid but deforms slowly compared to the drift time (not compared to a period of $\Omega$), it seems virtually certain that the second and third invariants would be conserved and the invariance of $\Phi$ would contribute nontrivial information. One would then have the picture of a rapidly rotating drift shell which deforms very slowly and about which the particle drifts on an intermediate time scale.

9. Summary and Conclusions

We have investigated some aspects of adiabatic theory in the regime where the $E \times B$ drift velocity is comparable with or larger than the gyro velocity. Previous work shows that such particles undergo five drifts in addition to the familiar $E \times B$, gradient, and line curvature drifts, have three more terms in the parallel equation of motion, and possess neither bounce $J$ nor flux $\Phi$ adiabatic invariants. The absence of conserved $J$ and $\Phi$ arises from the fact that a large $E \times B$ drift in general destroys the near periodicity of the bounce and drift motions.

The case of the rapidly rotating rigid magnetic field configuration is an exception to these conclusions, because a corotating particle then has nearly periodic motion. The equation of motion of a charged particle in a rapidly rotating asymmetric magnetosphere has a formal identity to the particle equation of motion in nonrotating static magnetic and electric fields. Because of this similarity, all previously developed adiabatic theory of guiding center motion can be used to produce analogous guiding center equations in the rotating reference frame. Using this analogy is an easy way to obtain the guiding center drift velocity and parallel differential equation of motion. The drift velocity (16) in the rotating frame consists of the expected field gradient and line curvature drifts plus centrifugal and Coriolis force induced drifts. The parallel equation of motion (27) contains mirror and centrifugal forces along the field line direction and zero order in gyroradius plus two first order terms.

The particle equation of motion in the rotating frame has an exact constant (5) of the motion equal to the particle kinetic energy in the rotating frame plus a centrifugal potential energy. If the particle moves further from the rotation axis, its rotating
TABLE 1. Summary of Equations for a Rigid Rotator

<table>
<thead>
<tr>
<th>Type of Equation</th>
<th>For Observer Rotating with the System</th>
<th>For a Nonrotating Observer Looking at the Rotating System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact particle equation of motion</td>
<td>( \left( \frac{d\mathbf{w}}{dt} \right)_c = \nabla \cdot \mathbf{p} + \frac{e}{mc} \mathbf{w} \times \left[ \mathbf{B}(t) + \frac{2mc}{e} \mathbf{\Omega} \right] )</td>
<td>( \overline{\mathbf{v}} ) ( \frac{d\mathbf{r}}{dt} = \frac{e}{mc} \left[ \mathbf{B}(t) + \frac{\mathbf{v}}{c} \right] \times \mathbf{B}(t) )</td>
</tr>
<tr>
<td>Exact constant of the motion</td>
<td>( H = \frac{1}{2} (\mathbf{w} \cdot \mathbf{p} + \mathbf{w} \cdot \nabla f) )</td>
<td>( \overline{\mathbf{v}} ) ( \frac{d\mathbf{r}}{dt} = \frac{e}{mc} \left[ \mathbf{B}(t) + \frac{\mathbf{v}}{c} \right] \times \mathbf{B}(t) )</td>
</tr>
<tr>
<td>Vlasov equation</td>
<td>( \frac{d\mathbf{f}}{dt} = \mathbf{w} \cdot \nabla f + \left[ \nabla \cdot \mathbf{p} + \frac{e}{mc} \mathbf{w} \times \left( \mathbf{B} + \frac{2mc}{e} \mathbf{\Omega} \right) \right] \cdot \nabla \mathbf{f} = 0 )</td>
<td>( \overline{\mathbf{v}} ) ( \frac{d\mathbf{f}}{dt} = \mathbf{w} \cdot \nabla f + \frac{e}{mc} \left[ \mathbf{B} + \frac{\mathbf{v}}{c} \right] \times \mathbf{B} \cdot \nabla \mathbf{f} = 0 )</td>
</tr>
<tr>
<td>Guiding center equation of motion</td>
<td>( \dot{\mathbf{R}}_c = \nabla \cdot \left( \frac{e}{mc} \mathbf{R}_c \times \left( \mathbf{B} + \frac{2mc}{e} \mathbf{\Omega} \right) \right) - \frac{M}{m} \nabla \mathbf{B} )</td>
<td>( \dot{\mathbf{R}}_c = \frac{e}{mc} \left( \mathbf{E} + \frac{\mathbf{R}_c \times \mathbf{B}}{c} \right) - \frac{M}{m} \nabla \mathbf{B} = \frac{e}{mc} \left( \mathbf{\Omega} \times \mathbf{R}_c \right) \times \mathbf{B} - \frac{M}{m} \nabla \mathbf{B} )</td>
</tr>
<tr>
<td>Guiding center drift velocity</td>
<td>( \dot{\mathbf{\epsilon}} = \frac{mc}{eB} \mathbf{\epsilon} \times \left( \nabla \cdot \mathbf{p} + \frac{M}{m} \nabla \mathbf{B} + u_i^2 \frac{\partial \mathbf{\epsilon}}{\partial s} + 2u_i \mathbf{\Omega} \times \mathbf{\epsilon} \right) )</td>
<td>( \dot{\mathbf{\epsilon}} = \dot{\mathbf{\epsilon}}_\perp + \frac{cE \times \mathbf{\epsilon}}{B} )</td>
</tr>
<tr>
<td>Guiding center parallel equation of motion</td>
<td>( \frac{du_i}{dt} = -\frac{M}{m} \frac{\partial \mathbf{B}}{\partial s} + \frac{\partial}{\partial s} \left( \frac{1}{2} \rho^2 \mathbf{\Omega}^2 - \frac{2Mc}{e} \mathbf{\Omega} \cdot \frac{\partial \mathbf{\epsilon}}{\partial s} \right) \ - \frac{mc}{eB} \mathbf{\epsilon} \times \left( \frac{u_i}{\partial s} \frac{\partial \mathbf{\epsilon}}{\partial s} + \mathbf{\Omega} \times \mathbf{\epsilon} \right) \cdot \nabla \left( \frac{MB}{m} - \frac{1}{2} \rho^2 \mathbf{\Omega}^2 \right) )</td>
<td>( \dot{u}_i = \frac{-M}{m} \frac{\partial \mathbf{B}}{\partial s} + \frac{\partial}{\partial s} \left( \frac{1}{2} \rho^2 \mathbf{\Omega}^2 + u_i \mathbf{\Omega} \cdot \frac{\partial \mathbf{\epsilon}}{\partial s} \right) )</td>
</tr>
<tr>
<td>Lowest order constant of the parallel motion</td>
<td>( K_c = \frac{1}{2} (u_i^2 - \rho^2 \mathbf{\Omega}^2) + \frac{MB}{m} )</td>
<td>( \frac{K_c}{m} = \frac{1}{2} (u_i - \rho \mathbf{\Omega} \cdot \mathbf{\epsilon})^2 - \frac{1}{2} \rho^2 \mathbf{\Omega}^2 + \frac{MB}{m} )</td>
</tr>
<tr>
<td>Rate of change of ( K_c )</td>
<td>( \frac{\dot{K}_c}{m} = -\frac{2Mc}{e} u_i \mathbf{\Omega} \cdot \frac{\partial \mathbf{\epsilon}}{\partial s} )</td>
<td>( \dot{K}_c = \frac{-2Mc}{e} (u_i - \rho \mathbf{\Omega} \cdot \mathbf{\epsilon}) \mathbf{\Omega} \cdot \frac{\partial \mathbf{\epsilon}}{\partial s} )</td>
</tr>
<tr>
<td>Gyro average total energy (kinetic + potential)</td>
<td>( H = \left( \frac{\mathbf{\Omega}^2}{2} - \frac{\rho^2 \mathbf{\Omega}^2}{2} \right) )</td>
<td>( H = \left( \frac{\mathbf{\Omega}^2}{2} \right) - \rho \mathbf{\Omega} \cdot \dot{\mathbf{R}}_c - \rho^2 \mathbf{\Omega}^2 )</td>
</tr>
<tr>
<td>Second adiabatic invariant</td>
<td>( J_c = m \int ds \left( \frac{1}{m} (K_c - MB) + \frac{\rho^2 \mathbf{\Omega}^2}{2} \right)^{1/2} )</td>
<td>same as for rotating observer</td>
</tr>
</tbody>
</table>

Here \( e \) stands for the total charge on the particle (\( e < 0 \) for negative charge); argument of \( \mathbf{B}, \mathbf{\epsilon}, \mathbf{p}, \mathbf{\Omega} \), etc., is \( \mathbf{R} \) (the guiding center position) in a guiding center equation, and \( \mathbf{r} \) (particle position) in a particle equation; and subscript \( c \) indicates as seen in corotating system. For example, \( \frac{d\mathbf{w}}{dt} \) seen from nonrotating system = \( \left( \frac{d\mathbf{w}}{dt} \right)_c + \mathbf{\Omega} \times \mathbf{w} \).
frame kinetic energy increases. This constant of the motion does not by itself put any limits on the particle energy increase or decrease. However, we have also shown in this paper that the second invariant is conserved in the rotating frame by the four drifts in (16), so that a particle slowly drifts around on its rapidly rotating drift shell and returns to its original field line. Thus there is no long-term energy change. Any energy change is periodic on the bounce and drift time scales. This same statement also holds for energy changes as observed by a nonrotating observer. This is in brief why all previous explicit calculations of particle energization in rigid rotators have failed to show any secular effect.

Finally, a Vlasov equation holds in the rotating system, again because of the analogy between equations (1) and (4).

Table 1 summarizes the most important equations from the point of view of both rotating and nonrotating observers, while the notation list defines the notation. Taken together, these two serve as a reference that can be used without delving into the details of the text.

**NOTATION**

- $B, E$: magnetic and electric fields;
- $\mathcal{A}, \mathcal{E}$: analog magnetic and electric fields;
- $\dot{\mathbf{\ell}}_1, \mathbf{\ell}_1$: particle position vector from origin (located on rotation axis $\Omega$);
- $v$: particle velocity;
- $v_0$: particle gyrovelocity;
- $\Omega$: angular rotation velocity of rigid rotator;
- $\rho$: distance from rotation axis;
- $\phi$: unit vector in azimuthal direction about rotation axis $\Omega$;
- $w$: particle velocity in rotating frame $= v - \rho \Omega \phi$;
- $\mathbf{R}_g$: guiding center position;
- $\mathbf{R}_{g,c}$: guiding center velocity seen from rotating frame;
- $\mathbf{R}_{g,k}$: guiding center velocity seen from nonrotating frame;
- $u_{\parallel}, u_{\perp}$: component of $\mathbf{R}$, perpendicular to $\mathbf{\ell}_1$;
- $v_{\parallel}, v_{\perp}$: guiding center acceleration seen in rotating frame;
- $s, \sigma$: distance along line of $B, \mathcal{A}$ from arbitrary zero on the line;
- $\partial/(\partial s), \partial/(\partial \sigma)$: guiding center position, respectively, of the enclosed quantity;
- $\overline{\epsilon}_1 \cdot \nabla_1 \mathbf{B} \cdot \nabla_1$: charge on particle (negative for negative charge);
- $\epsilon$: charge on particle (negative for negative charge);
- $\omega$: gyrofrequency $eB/mc$ (negative for negative charge);
- $M$: sum $M_0 + eM_1$ of first two terms of magnetic moment series;
- $J_2$: second (longitudinal adiabatic invariant for particle in a rigid rotator);
- $\Phi$: third or flux invariant;
- $\phi$: electrostatic potential;
- $T$: bounce period of guiding center;
- $H$: total (kinetic + potential) particle energy/m in the rotating system;
- $K_r$: constant of the parallel (to $B$) guiding center motion;
- $\mathcal{K}_r$: constant of the parallel (to $\mathcal{A}$) guiding center motion.

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**REFERENCES**


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