Abstract. Using the WKB approximation, we analyze the phase and energy structure of damped Alfvén wave structures in a magnetospheric field configuration. In the WKB limit the real part of the eigenfrequency is determined by the field line length while the damping rate is determined by the ionospheric conductivity. Expressions are worked linking the damping decrement to the ionospheric damping, and it is shown that the damping effect of each ionosphere adds linearly to the damping decrement. We end with a critique of the use of the WKB solution for Alfvénic wave structures and indicate where such solutions may be unreliable.

1. Introduction

Standing Alfvén waves are a well-known feature of the terrestrial magnetosphere. Being the natural eigenmodes of the system, they can be excited by a variety of mechanisms. Because they are eigenmodes of the system, their structure tells us about the system as well as about the waves themselves. Alfvén waves can directly reveal aspects of the coupling of the solar wind and the terrestrial magnetosphere. For example, energy from the solar wind causes magnetopause boundary motions (via Kelvin-Helmholtz instability or transient reconnection events or even simple buffeting) that excite compressional oscillations near the boundary. Some fraction of the energy tunnels deeper into the magnetosphere whereupon it can be converted into the field-guided Alfvén or transverse modes. The Alfvén mode signals are called field line resonances, and they are relatively localized standing waves trapped between northern and southern ionospheres of closed field lines.

Here we derive a simple analytic solution for standing Alfvén waves when the principal damping mechanism is absorption in the ionospheres at the feet of field lines. The work is a simple extension of solutions given by Newton et al. [1978], Allan and Knox [1979a, 1979b], Allan [1982], Ellis and Southwood [1983], and Southwood and Hughes [1983]. In particular, we extend Ellis and Southwood’s results for multiple reflections to include a first-order treatment of the inhomogeneity along the field by introducing the WKB form to describe the parallel phase propagation described by Southwood and Kivelson [2000]. Following Allan [1982], we also illustrate how the symmetry of the mode (with respect to the equator) is modified by asymmetry of the ionospheres.

The launch of the four Cluster spacecraft means that there will be new ways to measure structure within geomagnetic disturbances. This paper is a follow-up to the paper of Southwood and Kivelson [2000], which concentrated on the phase structure in the magnetosphere itself. The results in this paper have less relevance to interspacecraft measurements but may have a greater bearing on spacecraft-ionosphere or spacecraft-ground mapping.

2. Ionospheric Boundary Conditions

For the purpose of this paper we shall consider the major energy sink for standing waves to be the ionosphere. The ionosphere enters the study as a thin boundary layer at the field line feet which absorbs energy by Joule dissipation in the Pedersen currents that must flow transverse to the field there. The ionospheric absorption enters the standing mode problem through the boundary conditions. Let $s$ be the coordinate measured along the background field direction and let the ionospheres be at $s = 0$ and $s = a$.

The ionospheric boundary condition at $s = 0$ is calculated straightforwardly [see Hughes and Southwood, 1976]. To a good approximation, within the ionosphere, Ampère’s law gives

$$\mu_0 j = -\frac{\partial b}{\partial s},$$

where $j$ is the transverse (Pedersen) current and $b$ is the transverse magnetic field of the wave. Integrating through the “thin” ionosphere yields $\Delta b$, the change of $b$ across the ionosphere,

$$\Delta b = -\mu_0 \int_{s_0}^{s_1} ds' \sigma_P E = -\mu_0 \Sigma_P E,$$

where

\[
\Delta b = -\mu_0 \int_{s_0}^{s_1} ds' \sigma_P E = -\mu_0 \Sigma_P E,
\]

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where $\sigma_p$ is the Pedersen conductivity of the ionosphere,
$\Sigma_p = \int_{\text{ionosphere}} d\sigma_p$ is the Pedersen conductance, and $E$ is the
transverse wave electric field, constant across the ionosphere. The theory of Hughes and Southwood [1976] shows that the magnetospheric magnetic signal $b$ is completely shielded from the atmosphere below the ionosphere. (The ground signal comes from the magnetic signature of the (nondissipative) ionospheric Hall currents.) One deduces that the magnetospheric magnetic field at the top of the ionosphere is given by

$$b(s = 0) = \Delta b.$$  \hspace{1cm} (3)

The wave boundary condition at $s = 0$ is therefore

$$\frac{b(0)}{E(0)} = -\mu_0 \Sigma_{po},$$  \hspace{1cm} (4)

where $\Sigma_{po}$ is the height-integrated conductivity.

We take the other end of the field line at $s = \alpha$. There the sign
is reversed, and the height-integrated conductivity is $\Sigma_{pa}$. The
integration through the ionospheric layer is in the opposite sense
with respect to the direction of increasing $s$, and so the boundary
condition is

$$\frac{b(\alpha)}{E(\alpha)} = \mu_0 \Sigma_{pa}.$$  \hspace{1cm} (5)

### 3. Phase Structure in the WKB Approximation

Let us now calculate the eigenfrequencies for standing modes
in a magnetized plasma with density $\rho(s)$ and background field
$B(s)$ with the boundary conditions of (4) and (5). For the sake of
simplicity we shall assume that the background field and the
plasma density are symmetric about the equator. We write the
solution as the sum of oppositely traveling waves of electric
amplitude, $E(s,t)$ and $R E(s,t)$, respectively. $R$ is something like a
reflection coefficient in that it equals unity when the ionospheres
are perfectly reflecting. However, it will depend on the properties
of both ionospheres. Using the WKB approximation to describe
the spatial dependence, one has

$$E(s,t) = A \left[ \exp \left( -i\omega t - \int_0^s \frac{ds'}{V_A(s')} \right) + R \exp \left( -i\omega t + \int_0^s \frac{ds'}{V_A(s')} \right) \right],$$

where $V_A(s) = B(s)/(\mu_0 \rho)^{1/2}$ is the Alfvén speed and $A$ is the
maximum amplitude of the wave traveling in the $+s$ direction. For a loss-free system, $R = \pm 1$, where the sign depends on the
nature of the reflection at the boundaries at the field line feet. Perfectly conducting boundaries ($\Sigma_p \to \infty$) imply $R = -1$,
whereas $R = 1$ corresponds to low-conductivity ionospheres
($\Sigma_p \to 0$). Provided the boundaries are the same north and south,
solutions for these cases take the form

$$E(s,t) = 2A \exp(-i\omega t) \left\{ \frac{\cos}{\sin} \left( \omega \int_0^s \frac{ds'}{V_A(s')} \right) \right\} = 2A \exp(-i\omega t) \left\{ \frac{\cos}{\sin} \left[ \omega r(s) \right] \right\}.$$  \hspace{1cm} (7)

which defines $r(s)$, and where the sin and cos correspond to $R = -1$ and $R = +1$ and also the two different solutions correspond to
the field line being fixed or free, respectively. Also, from
Faraday’s law the magnetic field is given by

$$b(s,t) = A \frac{V_A}{\omega} \exp(-i\omega t) \left\{ \frac{\sin}{\cos} \left[ \omega r(s) \right] \right\}.$$  \hspace{1cm} (8)

Here

$$\omega r(s) = \omega \int_0^s \frac{ds'}{V_A(s')} = \varphi(s),$$  \hspace{1cm} (9)

where $\varphi(s)$ is the wave phase and the boundary conditions
require, in either case, $\varphi(\alpha) = n\pi$ and so

$$\omega \int_0^\alpha \frac{ds'}{V_A(s')} = n\pi \text{ or } \omega = \frac{n\pi}{r(\alpha)},$$  \hspace{1cm} (10)

where $n$ is an integer and $r(\alpha)$ is the wave travel time between
hemispheres.

As ionospheric solar illumination at high latitude can be
radically different at the two ends of a flux tube, one should note
the possibility of “quarter wave” solutions where one ionospheric
boundary is fixed ($E = 0$) and the other is free ($b = 0$). If the free
end is at $s = 0$, the solutions for the electric and magnetic field are

$$E(s,t) = 2A \exp(-i\omega t) \sin[\omega r(s)],$$
$$b(s,t) = 2A \exp(-i\omega t) \cos[\omega r(s)],$$  \hspace{1cm} (11)

Figure 1. Sketch of the variation of electric and magnetic fields with distance $s$ along $B$ for the fundamental
mode where there is a fixed end boundary condition at both ionospheres.
but now

\[ \varphi(a) = \left( \frac{n - 1}{2} \right) \pi, \]  

where \( n \) is an integer. The amplitude field variations of \( E \) and \( b \) for the case represented by (10) for \( n = 1 \) are plotted in Figure 1.

### 4. Effect of Ionospheric Damping

In a lossy system one must allow for damping of a normal mode. Hence we write \( \omega = \alpha + i \gamma \), where \( \alpha \) is the real frequency and \( \gamma \) is the damping rate. As before, the electric and magnetic fields are given by

\[ E(s,t) = A \left( \exp \left\{ -i \alpha t - \tau(s) \right\} + R \exp \left\{ -i \gamma t + \tau(s) \right\} \right), \]

\[ b(s,t) = \frac{A}{V_A(s)} \left( \exp \left\{ -i \alpha t - \tau(s) \right\} - R \exp \left\{ -i \gamma t + \tau(s) \right\} \right). \]

Applying the boundary conditions (4) and (5) at \( s = 0 \) and \( a \), one has

\[ \frac{V_A b}{E} \bigg|_{s=0} = -\nu \mu_a V_A^0 = \frac{1 - R}{1 + R}, \]

\[ \frac{V_A b}{E} \bigg|_{s=a} = \nu \mu_a V_A^0 = \frac{\exp(i \varphi(a)) - R \exp(-i \varphi(a))}{\exp(-i \varphi(a)) + \exp(i \varphi(a))}. \]

We define

\[ z_0 = \frac{1}{\mu_0 \nu \mu_a V_A^0}, \quad z_a = \frac{1}{\mu_0 \nu \mu_a V_A^0}. \]

Hence

\[ R + 1 \]

\[ 1 - R = -(z_0 - 1) \]

\[ R \exp[-i2 \varphi(a)] + 1 \]

\[ 1 - R \exp[-i2 \varphi(a)] = z_a, \]

\[ R = \frac{z_a + 1}{z_a - 1} \]

\[ R \exp[-i2 \varphi(a)] = \frac{z_a - 1}{z_a + 1}. \]

Eliminating \( R \), one has an equation for \( \varphi(a) \), which determines the allowed values of \( \alpha \).

\[ \exp(i2 \varphi(a)) = \left( \frac{z_a + 1}{z_a - 1} \right) \left( \frac{z_a + 1}{z_a - 1} \right). \]

Hence one may then write

\[ i2 \varphi(a) = \ln \left( \left( \frac{z_a + 1}{z_a - 1} \right) \left( \frac{z_a + 1}{z_a - 1} \right) \right). \]

The logarithm of a complex number is, in general, multivalued. So \( \ln[A] = \ln[A] + i2n\pi \) for integer \( n > 0 \), whatever the value of \( A \). Furthermore, for any negative quantity, \(-A\), say, \( \ln[-A] = \ln[A] + i(2n+1)\pi \). We thus have two alternatives, which depend on the sign of \((z_0 - 1)(z_a - 1)\):

Case 1, \((z_0 - 1)(z_a - 1) > 0\)

\[ i2 \varphi(a) = i2n\pi + \ln \left( \left( \frac{z_a + 1}{z_a - 1} \right) \left( \frac{z_a + 1}{z_a - 1} \right) \right). \]

Case 2, \((z_0 - 1)(z_a - 1) < 0\)

\[ i2 \varphi(a) = i(2n+1)\pi + \ln \left( \left( \frac{z_a + 1}{z_a - 1} \right) \left( \frac{z_a + 1}{z_a - 1} \right) \right). \]

where in either case, \( n \) is any positive integer.

There is also a singular case when \((z_0 - 1)(z_a - 1) = 0\). In this case, where either \( z_0 = 1 \) or \( z_a = 1 \), the conductance of one ionosphere matches the magnetospheric impedance \( 1/V_A \mu_a \) precisely. Then in this WKB approximation there is complete absorption of an Alfvén wave incident on the ionosphere at one or the other end of the field line. No reflection can occur there, and any signal set up on the field line is immediately absorbed within an Alfvén travel time. Standing signals cannot occur in such circumstances.

Now recalling that

\[ \varphi(a) = i2 \alpha \tau(a) \]

and separating real and imaginary parts of \( \alpha \),

\[ \varphi(a) = (\alpha - i \gamma) \tau(a), \]

one finds that the allowed standing wave frequencies for case 1 are given by

\[ \omega \tau(a) = \omega_b \left( \frac{1}{V_A} \right)^2 = n \pi \quad \text{and} \quad \omega_b = \frac{n \pi}{\tau(a)}. \]

Interestingly, there is no change in the real part of the frequency in making the transition from loss-free to finite loss case as this form applies to either both ends loss-free or both ends fixed. However, in the more asymmetric circumstance of case 2 where one end is approximately fixed and the other end is approximately free, one has

\[ \omega_b \tau(a) = \omega_b \int_0^a \frac{ds'}{V_A(s')} = \frac{(2n - 1) \pi}{2}, \]

\[ \omega_b = \frac{(2n - 1) \pi}{2 \tau(a)}. \]

Cases 1 and 2 evidently correspond to "half-wave" and "quarter-wave" cases of the loss-free case [Allan and Knox, 1979b; Allan, 1983].

In both cases 1 and 1, the damping decrement is given by

\[ \gamma = \frac{1}{2 \alpha} \ln \left( \left( \frac{z_a + 1}{z_a - 1} \right) \left( \frac{z_a + 1}{z_a - 1} \right) \right). \]

Similar expressions have been derived previously by several authors, such as Allan and Knox [1979b], Southwood and Hughes [1983], and Ellis and Southwood [1983]. Note that the real part of the eigenfrequency in (26) or (27) is controlled by the length of the field line and that only the damping is affected by the ionospheric conductivity. Moreover, as one can write in all cases

\[ \gamma = \frac{1}{2 \alpha} \ln \left( \left( \frac{z_a + 1}{z_a - 1} \right) \left( \frac{z_a + 1}{z_a - 1} \right) \right), \]

one can see that the effect of each ionosphere can be separated out. Indeed, one sees that the contributions of each ionosphere to
the damping decrement are strictly linearly additive and independent of harmonic number $n$.

Note from (19) that (29) can also be written in terms of the reflection coefficients at each ionosphere:

$$\gamma = \frac{1}{2 \tau(a)} \ln |R_0| + \ln |R_a| \right), (30)$$

where $R_0 = R$ and $R_a = R' \exp \left[-2\varphi(a)\right]$. It is also useful to note that for $z_0, z_a \ll 1$ (a case 1 situation), one has

$$\gamma = \frac{1}{\omega_R} \left( \frac{1}{\mu_0 \Sigma_{pl} V_A^0} + \frac{1}{\mu_0 \Sigma_{pl} V_A^a} \right). \right) (31)$$

Similarly, when the ionospheric conductivities are low enough that $z_0, z_a \gg 1$ (again a case 1 situation)

$$\gamma = \frac{1}{\omega_R} \left( \frac{1}{\mu_0 \Sigma_{pl} V_A^0} + \frac{1}{\mu_0 \Sigma_{pl} V_A^a} \right). \right) (32)$$

In the case where the signal adopts quarter wave-structure (a case 2 situation), if $z_0 \ll 1$ and $z_a \gg 1$, one has

$$\gamma = \frac{2}{\omega_R} \left( \frac{1}{2n-1} \pi \right) \left( \frac{1}{\mu_0 \Sigma_{pl} V_A^0} + \frac{1}{\mu_0 \Sigma_{pl} V_A^a} \right). \right) (33)$$

5. Illustrations of Wave Structure and Damping

5.1. Symmetric Ionospheres with Electric Near Node at the Ionosphere ($z_0, z_a \ll 1$)

The significance of the results discussed here is most simply illustrated by example. We first consider a standing oscillation in a symmetric system (i.e., where the ionospheric conductivities are the same at each end of the field line).

Equations (26) and (27) show that the real part of the frequency does not change when the ionospheric conductivity changes by a small amount, consistent with a WKB approach. It follows that in this limit one should also be able to write the spatial amplitude of the electric and magnetic fields in the form

$$E(s) = E_0(s) + dE(s) b(s) = b_0(s) + db(s), \right) (34)$$

where the subscript zero indicates the corresponding field in the absence of damping and where the "perturbation" fields, $dE(s)$ and $db(s)$, are proportional to $\gamma$. We shall assume for the moment that for $\gamma = 0$ the ionospheric boundary conditions are fixed end conditions, i.e.,

$$E_0(a) = E_0(0) = 0. \right) (35)$$

Now let us consider the solution as an expansion in the small parameter $\gamma/\omega_R$. The phase is also separable into a part independent of $\gamma$, $\varphi_0(s)$, and a linear part proportional to $\gamma$, which we write as $d\varphi(s)$. We now choose $d\varphi(s)$ so that it is measured from zero at the field line midpoint, $s = a/2$. We redefine $\varphi(s)$ to be zero at the equator, $s = a/2$, by writing

$$\varphi(s) = \frac{\omega_R}{\gamma} \int_{a/2}^{s} \frac{ds'}{V_A(s')} = \varphi_0(s) + d\varphi(s). \right) (36)$$

This means that for $s < a/2$ the phase integral $\varphi_0$ is negative. The phase integral is thus defined to increase monotonically from a negative value in the ionosphere at $s = 0$ to zero at the equator after which it is positive. It follows that

$$\varphi_0(s) = \omega_R \int_{a/2}^{s} \frac{ds'}{V_A(s')} d\varphi(s) = i\gamma \int_{a/2}^{s} \frac{ds'}{V_A(s')} \right) (37)$$

By writing $E(s)$ in the form

$$E(s) = A(s) F[\varphi(s)] \right) (38)$$

we deduce (see equations (34) and (37)) that

$$dE(s) = -i \gamma \omega_R b_0(s) A(s) d\varphi(s). \right) (39)$$

where

$$\frac{dF}{d\varphi} = \frac{V_A(s) d}{ds} \left[F[\varphi_0(s)]\right] = \frac{V_A(s)}{\omega_R} \frac{d}{ds} \left[\frac{E_0(s)}{A(s)}\right]. \right) (40)$$

The relationship between $E_0$ and $b_0$ is illustrated in Figure 1 for the fundamental for the fixed end ionospheric boundary conditions. Using the notation of (38) and putting $d/d\varphi = V_A(s)/\omega_R[d/ds]$, one may write

$$b_0(s) = \frac{1}{i\omega_R} \frac{d}{ds} \left[A(s) F[\varphi_0(s)]\right] \right) (41)$$

which can be reexpressed as

$$A(s) \frac{d}{d\varphi} \left[F[\varphi_0(s)]\right] = i V_A(s) b_0(s) - \frac{V_A(s)}{\omega_R} \frac{d}{ds} \left[\frac{E_0(s)}{A(s)}\right], \right) (42)$$

From (43) and (39) it follows that

$$dE(s) = -i \gamma \omega_R b_0(s) A(s) \frac{dF}{d\varphi} \right) (43)$$

Now let us consider the boundary condition at $s = a$ given in (16). Equation (35) requires that $E_0(a) = 0$. From (45), $dE_H(a) = 0$, and using (16), one has
\[ \frac{E(a)}{b(a)} = \frac{dE(a)}{d(b(a))} = \frac{1}{\mu_0 \sum p_{bo}}. \]  

Substituting from the form (44) gives

\[ dE(a) = dE(b(a)) = \gamma \varphi_0(a) [V_A(a)b_0(a)]. \]  

Hence to lowest order in \( \gamma \),

\[ \frac{1}{\mu_0 \sum p_{bo}} = \gamma \varphi_0(a) V_A(a). \]  

Noting, from

\[ \varphi_0(a) = \omega_R \int \frac{ds'}{V_A(s')} = \frac{\omega_R}{2} \int_0^a \frac{ds'}{V_A(s')} = \frac{n\pi}{2}, \]

one finds

\[ \frac{\gamma}{\omega_R} = \frac{2}{n\pi \mu_0 \sum p_{bo} V_A(a)}. \]  

Equation (50) is consistent with the general expression given in (31) in the limit where \( \gamma/\omega_R \) is small in the case where \( z_0 \) and \( z_1 \) are equal and < 1.

At points above the ionospheres, there are four terms proportional to \( \gamma \) that contribute to the Poynting flux, \( S(s) \),

\[ S(s) = \{ dE_B(s)b_0(s) + E_0(s)[db_0(s) + \varphi_0(s)] + dE_B(s) b_0(s)/\mu_0 \}. \]

where \( db_0(s) \) and \( dE_B(s) \) are terms in \( b(s) \) proportional to \( \gamma \) and to \( E_0(s) \) and \( b_0(s) \), respectively. The latter quantities can be calculated analogously to \( dE_B(s) \) and \( dE_0(s) \) but using the plasma equation of motion (\( \partial b_0/\partial s = -i\omega E_0 V_A^2 \)) in place of Faraday's law (equation (41)).

For the fundamental, Figure 3 illustrates how \( dE_0(s) \) contributes to the Poynting flux along the field. All the terms in (51) other than the first term are precisely zero at the ionosphere because of the boundary condition (35) assumed initially. In the fundamental the terms proportional to \( E_0(s) \) are never important in the case considered here. However, for higher harmonics, in which there are inevitably nodes (zeros) in \( b_0 \), the other terms are important and provide a continuity in energy flux through the nodes of \( b_0 \).

5.2. Nonsymmetric Ionospheres

Once one recognizes that the term \( \varphi_0(s) \) controls the symmetry of the energy flux to the ionospheres, one can see how to generalize to the situation where ionospheres are not symmetrical. Recall that we constructed \( \varphi_0(s) \) to be antisymmetric about the equator. If the ionospheres are not symmetric, then one requires the wave perturbations to satisfy separate boundary conditions (4) and (5). From (32) one must have

\[ \frac{\gamma}{\omega_R} = \frac{1}{n\pi \mu_0 \sum p_{bo} V_A(0)} + \frac{1}{n\pi \mu_0 \sum p_{bo} V_A(a)}. \]

The two different boundary conditions at \( s = 0 \) and \( a \) can be satisfied simultaneously by moving the point at which the phase is defined as zero. Thus we redefine \( \varphi(s) \) so that it is measured from \( s = s_0 \):

\[ \varphi(s) = \omega \int \frac{ds'}{V_A(s')} = \varphi_0(s) + d\varphi(s), \]

\[ \varphi_0(s) = \omega_R \int \frac{ds'}{V_A(s')} \quad d\varphi(s) = -i\gamma \int \frac{ds'}{V_A(s')} \ . \]

Much as before, the phase integrals \( \varphi_0(s) \) and \( id\varphi(s) \) are deemed negative for \( s < s_0 \).

With the revised definition of \( s_0 \), (52) remains valid, but now both the boundary conditions (4) and (5) can be applied:

\[ \text{Legend: } E_0 \quad \text{ib}_0 \quad idE_b \]

\[ s = 0 \quad s = a \]

Figure 2. As in Figure 1, for the case of weakly absorbing boundaries corresponding to symmetric ionospheres. The dotted curve is the component of the electric field in phase or in strict antiphase with \( b \).
Bearing in mind that $\phi_0(0)$ is negative, one deduces that

$$\frac{\phi_0(0)}{\phi_0(a)} = \frac{\Sigma_{pa} V_a(a)}{\Sigma_{pa} V_a(0)}.$$

Furthermore,

$$\frac{\gamma}{\omega_R} [\phi_0(a) - \phi_0(0)] = \frac{1}{\mu_0 \Sigma_{pa} V_a(0)} + \frac{1}{\mu_0 \Sigma_{pa} V_a(a)}$$

and

$$[\phi_0(a) - \phi_0(0)] = \omega_R \left[ \int_0^a \frac{ds'}{V_a(s')} - \int_0^a \frac{ds'}{V_a(s')} \right]$$

$$= \omega_R \int_0^a \frac{ds'}{V_a(s')} + \omega_R \int_0^a \frac{ds'}{V_a(s')} = n \pi.$$

Thus one finds

$$\frac{\gamma}{\omega_R} = \frac{1}{n \pi} \left( \frac{1}{\mu_0 \Sigma_{pa} V_a(0)} + \frac{1}{\mu_0 \Sigma_{pa} V_a(a)} \right)$$

consistent with (31), as required. Each ionosphere independently drains energy from the wave, and as the ionosphere with the highest conductivity absorbs the least energy, the null point (which is also the point from which we measure $\phi$) is shifted from the equator toward that ionosphere. From (58) one deduces that $s_0$ is given by

$$s_0 = \left( \frac{a}{2} + \frac{n \pi V_a(a/2) \Delta(V_a \Sigma_p)}{4 \omega_R V_a \Sigma_p} \right),$$

where $\Delta(V_a \Sigma_p) = V_a(a) \Sigma_{pa} - V_a(0) \Sigma_{pa}$ and $V_a \Sigma_p = V_a(a) \Sigma_{pa}$ or $V_a(0) \Sigma_{pa}$, the distinction being irrelevant in the approximation assumed. If $V_a$ does not vary along $B$ as assumed by Allan [1982], (63) reduces to $s_0 = a/2 [1 + (\Sigma_{pa} - \Sigma_{pa})/(\Sigma_{pa} + \Sigma_{pa})]$, which is Allan's expression. The point $s = s_0$, where the energy flux changes sign, is called the null point by Allan [1982].

It is important to note that for nonsymmetric ionospheres there must be a Poynting flux through the equator. It follows that the equator will no longer be a strict node for the magnetic field and that there will be a finite magnetic signal there (see Figure 4). This may have important consequences for attempts to infer wave symmetry from the positions of the nodes of the wave magnetic field.

5.3. Symmetric Ionospheres with a Magnetic Near Node at the Ionosphere

A similar analysis can be provided for the situation where $z_0$, $z_0 > 1$ and the ionospheric boundary conditions are closer to a free end condition ($b = 0$). In this case the approximate solutions again yield a Poynting flux containing terms as in (52):

$$S(s) = \left[ dE_a(s) + dE_a(s) \right] b(s) + E_a(s) [db_a(s) + db_a(s)] / \mu_0.$$
However, now the term that is nonzero at the ionosphere (at \( z = a \), for example) is

\[
S(a) = \frac{E_0(a)db(a)}{\mu_0}.
\]  

### 6. Phase Variation Along the Field

The damped WKB standing wave solutions given in this paper also allow analysis of the variation of signal phase along \( B \). In the loss-free cases the relative phase of fields varies along the field, if at all, in steps of \( \pi \). As has been noted since the work of Newton et al. [1978] and Allan [1982], once damping is included, the phase of the signals varies continuously along the field. The recent work of Southwood and Kivelson [2000] points out that the spatial phase variation is directly related to energy flux; there is always an apparent phase motion along \( B \) in the direction of energy transport.

The spatial dependences of \( E(s,t) \) and \( b(s,t) \) are given by

\[
E(s) = A(s) \exp[i\alpha(s)]
\]

(see equations (13) and (14)). \( E(s) \) can be rewritten in the form

\[
\tan \alpha(s) = \frac{2 \tan[\omega_0 \tau(s)]}{2 + 2\gamma \tau(s)} \tan[\omega_0 \tau(s)].
\]  

Similarly,

\[
\tan \beta(s) = \frac{\varepsilon + 2\gamma \tau(s)}{2} \tan[\omega_0 \tau(s)].
\]

An example showing the variation of \( \beta(s) \), i.e., the phase angle of \( b(s) \), is plotted in Figure 5 for the fundamental standing mode where the ionospheres are not symmetric. In Figure 6 we sketch the phase behavior for the second harmonic in similar circumstances. Note that in each case the sense of phase variation along the field reverses at the null point. This is consistent with the fact that phase variation in both electric and magnetic fields must have the same sense of variation as the energy flux does (which reverses at the null point).

Note how the phase changes in a step-like manner that becomes more abrupt as the damping decreases. Many more similar examples, including higher harmonics and asymmetric structures, can be found in the plots given by Allan [1982].

### 7. Summary and Some Thoughts on the WKB Approach

The purpose of this paper has been to illustrate in a simple way the effects on pulsation structure that can be produced by inhomogeneity and asymmetric ionospheric boundary conditions. The important effects, which can almost certainly be carried forward to more complex (and realistic) models of the magnetosphere, concern the manner in which the electric and magnetic field structures change when the northern and southern ionosphere conductivities are finite and possibly of different magnitudes. In the limiting case where the ionospheric boundary condition corresponds to almost perfect conductivity, a small electric field, in phase with the wave magnetic field (and in quadrature with the zeroth-order electric field), provides the means whereby energy is carried along the field. The phase of the electric field is thus modified at each point on the field line. As we have shown elsewhere [Southwood and Kivelson, 2000], the result is that phases differ between high altitudes and the ground or ionosphere. Once the ionospheres are allowed to be asymmetric, the Poynting flux also becomes asymmetric with respect to the field equator. As we showed and as was first outlined by Allan [1982], there is a null point, located away from the equator in the hemisphere containing the most reflecting ionosphere, where the Poynting flux is zero.

![Figure 5. Example of the phase variation of the magnetic component, \( \beta(s) \), along \( B \) in the fundamental for different ionospheric conductivities at the two ends of the flux tube. The change of sense of phase variation takes place at the null point, where the Poynting flux reverses.](image)
If ionospheric damping is included, our calculations reveal that to first order the real part of the eigenfrequency is unchanged with respect to the undamped case. In other words, the real part of the eigenfrequency is determined only by the length of the field line and the distribution of Alfvén speed along the field. The damping decrement is found to be proportional to the sum of the reciprocals of the ionospheric conductances when the ionosphere is highly conducting. In the opposite extreme, where the ionospheric boundary condition is close to "free end", the damping decrement is proportional to the sum of the conductances. The decrement itself is not proportional to harmonic number $n$. One finds $\gamma/\omega_o \sim 1/n$; thus the "$Q$" factor is proportional to $n$. Asymmetric ionospheres do not change this result.

Our WKB-based results show that the damping becomes infinite when the ionospheric impedance precisely matches the magnetospheric impedance. The point where this occurs marks the transition between "free end" and "fixed end" boundary conditions. Comparing with the general calculations given by Newton et al.'s [1978] full solutions where maximum damping is obtained as a function of ionospheric conductance, one sees that they found that the growth rate maximized but did not become infinite. Thus one concludes that our WKB results can only be a guide to behavior in a more complex model.

Despite its approximate nature, the WKB approach serves as the basis of simple analysis that reveals interesting aspects of the standing Alfvén wave structure. In particular, it provides useful insight into the influence of the ionospheric boundary conditions on magnetospheric wave structure. The results on Poynting flux and the parallel phase illustrated here probably justify the use of the approach.

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