INTRODUCTION AND THEORY

In this paper we present a calculation of long-period, ultra-low-frequency (ULF) magnetospheric pulsations that uses a realistic earth's magnetic field geometry rather than a simple dipole field. The calculation pertains to transverse standing wave oscillations, whether they are Pc 3, 4, or 5 continuous dipole field. The calculation pertains to transverse standing wave oscillations in the same period range. The presence of standing wave oscillations, whether they are Pc 3, 4, or 5 continuous dipole field. The calculation pertains to transverse standing wave oscillations in the same period range. The presence of standing Alfvén wave resonances in the earth's magnetic field lines has been clearly demonstrated by conjugate point and spacecraft observations [see, e.g., Nagata et al., 1963; Sugita and Wilson, 1964; Van Chi et al., 1968; Lanzerotti et al., 1972, Lanzerotti and Fukunishi, 1972; Kokubun et al., 1976; Singer and Kivelson, 1979]. Several mechanisms have been suggested to generate these waves, but regardless of the method of generation, intrinsic interest in this fundamental magnetohydrodynamic plasma process and the possibility of using the waves to diagnose magnetospheric properties make it worthwhile to model these standing wave oscillations.

In the following presentation the basic MHD equations are used to derive an equation for the period and amplitude of oscillations in a dipole magnetic field, whereas only WKB approximate solutions have been used in more general geometries. We have developed a solution of the decoupled equations that includes both a general magnetic field geometry and the effects of density and mass composition. The aim of this paper is to isolate and examine the effect on eigenfrequencies of only the field geometry by keeping density constant along all field lines. We review the diurnal variations in wave period predicted on the ground and in space by using the recent Olson-Pfitzer magnetospheric magnetic field model in our solution. For example, on the ground at 67° magnetic latitude the diurnal variation in period caused by field geometry is larger than a factor of 2. At 6.6 Re, where the dipole field line from 67° crosses the magnetospheric equator, there is negligible diurnal variation in period. Significant diurnal variations in period (±10%) at fixed radial distance in the equatorial plane in space occur only at distances ≤10 Re. Knowledge of the field geometry is shown to be important for the determination of mass density in space from ground pulsation observations. We discuss the impact of our results in interpretation of experimental data.

Faraday's law:
\[ \nabla \times E = \partial \mathbf{B} / \partial t \]  
Ohm's law with infinite conductivity (or the frozen in flux condition):
\[ E = -\partial \xi / \partial t \times B_0 \]  
Ampere's law:
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \]  
and the momentum equation:
\[ \rho \left( \partial^2 \xi / \partial t^2 \right) = j \times B_0 \]  
where \( B_0 \) is the unperturbed background magnetic field; \( E \) and \( \mathbf{b} \) are the wave perturbation electric and magnetic fields, respectively; \( \xi \) is the plasma (or field) displacement; \( j \) is the current; and \( \rho \) is the plasma mass density. Equation (4) is an approximate equation. It does ignore several effects of there being a background current in the plasma. The appendix discusses the validity of (4). The wave equation satisfied by the field displacement \( \xi \) is

\[ \left( \partial^2 (B_0 \times \xi) / \partial t^2 \right) = \nabla \times \nabla \times (B_0 \times \xi) \]  
where \( \nabla \times \nabla \times (B_0 \times \xi) \) is a term that depends on the wave electric or magnetic field, has been used by numerous authors as a starting point for examining long-period ULF waves in the earth's magnetic field [Westphal and Jacobs, 1962; Dungey, 1963; Radoski and Carovillano, 1966; Cummings et al., 1969; Orr and Mathew, 1971; Radoski, 1974]. In a uniform plasma and magnetic field, (1)-(4) can be used to describe two magnetohydrodynamic wave modes, called the shear Alfvén or transverse mode, and the fast mode. An important feature distinguishing these two wave modes is that the perturbation magnetic field is strictly transverse to the ambient field for the transverse mode, while the fast mode has a component along the ambient field. In other words the fast mode can compress the field, whereas the transverse mode only bends the field. Derivations are given, and further characteristics of the wave modes are discussed in detail in Dungey [1967, 1968]. In particular, in the transverse mode wave, en-
ergy is guided along $B$, whereas in the fast mode, energy moves in the direction of the wave vector $k$, which can make any angle with $B$.

The introduction of a nonuniform magnetic field, such as a dipole field, makes it much more difficult to find solutions to the wave equation (5). Additional terms involving the spatial derivatives of $B$ appear in the equations. The result is that the fast and transverse modes are coupled. The coupled equations have not been solved, even in a dipole background field [Lanzerotti and Southwood, 1979].

Since it has not been possible to solve the coupled equations, approximations have been made to find the eigenfrequencies for the waves in special situations that decoupled the equations [Radoski and Carovillano, 1966; Dungey, 1967, 1968; Cummings et al., 1969; Orr and Matthew, 1971; Orr, 1973, Lanzerotti and Fukunishi, 1974]. Three of the cases for a dipole background field have been summarized by Orr [1973], who considers hydromagnetic waves with a longitudinal variation given by $e^{i\omega t}$, where $\phi$ is longitude and $m$ is the azimuthal wave number. Assuming an axisymmetric disturbance (putting $m = 0$), one finds the equations decouple to give signals with strictly toroidal and poloidal magnetic perturbations. The axially symmetric toroidal case is a transverse Alfvén mode with torsional oscillations ($b_1$) of an entire magnetic shell. The Poynting flux is aligned with $B_0$, so this case is like a pure shear Alfvén mode. The wave equation depends only on how the signal varies along $B_0$, and as a result, eigenmodes with wavelengths comparable to flux tube length have eigenfrequencies that are different for different shells. The axially symmetric poloidal case corresponds to the fast mode with magnetic oscillations in a meridian plane ($b_2$ and $b_3$) and represents alternate symmetric compressions and expansions of the entire magnetosphere. The Poynting flux is directed across field lines. The third case where decoupling is possible is the highly asymmetric poloidal mode because of its strong interaction with the solar wind. In particular, on high-latitude magnetic field lines, any model of the earth's magnetic field must allow for the field induced by the interaction with the solar wind. In an early attempt to consider the effect of solar wind compression of the earth's dipole field, Westphal and Jacobs [1962] solved the toroidal mode wave equation in a cylindrical geometry by using a compressed dipole field. Their model predicted that wave periods observed from a fixed geomagnetic latitude would decrease as the dipole was compressed. Of course, the solar wind does not simply compress the earth's field uniformly at all local times, and recently, more accurate representations of the earth's magnetic field have become available. Warner and Orr [1979] used the Mead and Fairfield [1975] magnetic field model and solved for wave periods by using the WKB approximation to the toroidal wave equation. Their results included local time variations, but the WKB approximation is not valid when the wavelength of the oscillation is comparable to the scale size of the system, and is particularly poor for the fundamental mode oscillation of a field line.

Rather than use an approximate WKB solution or the uncoupled toroidal and poloidal modes wave equations, we derived a single exact linear wave equation which can be solved in a dipole field for both toroidal and poloidal mode oscillations. In addition, unlike the Cummings et al. [1969] and Orr and Matthew [1971] equations, which were for a strictly dipolar field geometry, our wave equation can also be solved in a nondipolar field geometry. Again, we have the constraint that the waves must be strictly transverse, without any perturbation along the ambient field direction. However, in our formulation the solution to the wave equation can be found for any linear polarization direction perpendicular to the ambient field. It is also worth noting that none of the models we have been discussing, including ours, solves the self-consistent problem which includes the effect of a field line perturbation on all field lines in its vicinity.

To examine the oscillation of isolated field lines in a more
general field, first consider two adjacent field lines separated at some point along the normal to one by distance $\delta_s$. At any other point along the field line, we can define $h_s$ by requiring the normal separation to be

$$h_s \delta_s$$

If we then write the normal unit vector between the field lines as $\hat{\alpha}$, we can put

$$\nabla \alpha = \hat{\alpha} / h_s$$

Now consider a small displacement in the $\hat{\alpha}$ direction, $\xi_s$. From (1) and (2) the displacement produces a magnetic perturbation

$$b = \nabla \times (\xi_s \hat{\alpha} \times B_0)$$

Now

$$b \cdot \nabla \alpha = \nabla \alpha \cdot \nabla \times (\xi_s h_s \hat{\alpha} \times \nabla \alpha \times B_0)$$

$$= \nabla \cdot (\xi_s h_s \nabla \alpha^2 \hat{\alpha})$$

because $\nabla \alpha$ has been defined as perpendicular to $B_0$. Now also

$$|\nabla \alpha|^2 = h_s^{-2}$$

and so

$$b \cdot \nabla \alpha = (b_s / h_s) = B_0 \cdot \nabla (\xi_s / h_s)$$

Now in the same manner as Cummings et al. [1969], assume that the displacement is made with negligible field compression. The magnetic perturbation $b_s$, using (3) and (4), produces a force in the momentum equation

$$\mu_0 \partial (\xi_s / h_s) / \partial t = (\nabla \times b_s \hat{\alpha} \times B_0$$

$$= (\nabla \times b_s h_s \nabla \alpha) \times B_0$$

$$= (\nabla b_s h_s \times \nabla \alpha) \times B_0$$

$$= (B_0 \cdot \nabla b_s h_s) \nabla \alpha$$

$$\mu_0 \partial (\xi_s / h_s^2) / \partial t = (1 / h_s^2) (B_0 \cdot \nabla b_s h_s)$$

Taking this with (6) gives a wave equation

$$\mu_0 \partial^2 (\xi_s / h_s) / \partial t^2 = 1 / h_s^2 B_0 \cdot \nabla \{h_s^{-2} (B_0 \cdot \nabla (\xi_s / h_s))\}$$

To solve (8) numerically, we assume a time dependence of the form $e^{i \omega t}$. Writing $ds$ for the increment of length along the magnetic field direction at any point, we can rewrite (8) as a second-order differential equation

$$\partial^2 \left( \xi_s / h_s \right) / \partial t^2 + \left( \partial \left( \xi_s / h_s \right) / \partial s \right) \ln(h_s^2 B_0) \frac{\partial}{\partial s} \left( \frac{\xi_s}{h_s} \right) + \mu_0 \rho \omega^2 \frac{B_0^2}{B_s^2} \frac{\xi_s}{h_s} = 0$$

Once $\xi_s$ is determined, we can find $b_s$ from (6), which we rewrite as

$$b_s = h_s B_0 \frac{\partial}{\partial s} \left( \frac{\xi_s}{h_s} \right)$$

The wave electric field is given by

$$E_\alpha = -i \omega \xi_s B_0$$

where

$$\beta = \frac{B_s}{|B_0|} \times \hat{\alpha}$$

and the plasma velocity by

$$u_s = i \omega \xi_s$$

Any initial polarization or perturbation direction $\hat{\alpha}$ perpendicular to $B_0$ can be chosen. In practice, the geometrical factors $h_s$ are determined by first taking a perturbation direction at a particular point, e.g., the equator, and then by field line tracing in whatever field model is being used. The factor $h_s$ is proportional to field line separation and varies along $B_0$ (i.e., varies with $s$).

**ANALYSIS**

With the formulation developed above, we can solve for Alfvén eigenfrequencies in an arbitrary field geometry, and in particular in one that accurately represents the earth's magnetic field. First, we can compare (9) with the decoupled equations used by Cummings et al. [1969] if we solve (9) in a dipole field and use the same assumptions outlined by Cummings et al. The initial step in applying our method is to calculate the scale factors $h_s$. In Cummings et al. the scale factors are uniquely specified by the dipole field, whereas in our method the $h_s$'s must be determined and depend on the field geometry used. In a dipole field the separation between two field lines in a meridian is proportional to $(r \theta \sin \theta)^{-1}$, and the separation between two lines on the same magnetic shell is proportional to $r \theta$, where $\theta$ is colatitude and $r$ is radial distance from the dipole. Thus with $h_s = (r \theta \sin \theta)^{-1}$ in (9) we obtain the guided poloidal mode equation (b, $\xi_s$ in the meridian), and with $h_s = r \theta$ we obtain the toroidal equation (b, $\xi_s$ out of the meridian). Using these forms, we tested our numerical procedure and rederived the solutions for a dipole field, obtaining agreement with Cummings et al.'s results.

With confidence established in the application of wave equation (9) we introduced a more realistic magnetic field model in place of the dipole. The field from magnetospheric sources was calculated by using the Olson-Pfitzer model [W. P. Olson and K. A. Pfitzer, unpublished manuscript (preprint), 1977; Walker, 1979]. A dipole was used to model the earth's intrinsic field.

The procedure to calculate the standing wave periods is identical to that outlined above, except that through any point in space the field line in the Olson-Pfitzer model is substituted for the dipole field line. It is important to point out that the procedure used here for calculating the $h_s$'s limits us to examining transverse waves polarized in the direction radially outward from the earth and perpendicular to that direction. In the case of a dipole field these directions are considered poloidal and toroidal, respectively. Another way of looking at it is that surrounding the Olson-Pfitzer field line we consider a flux tube that intersects the equatorial plane in a rectangle with sides parallel and perpendicular to the radial direction. The $h_s$'s are then calculated by keeping flux constant in the tube as we progress up the field line in the calculation. There is a further implicit assumption in our method. Namely that locally all field lines are swept back out of the meridian to the same degree. If this is not so, the direction of polarization of the signal might be expected to vary as one moved along the field. We are currently developing procedures to probe this effect.

The Olson-Pfitzer model includes magnetic field contributions from the distributed quiet-time ring current, magnetopause current, and tail current systems. The model is designed to represent the observations of Sugiura and Poros [1973] and is limited to field lines that cross the equator inside of 15 $R_E$. Although the Olson-Pfitzer model can be used for all dipole
The ionosphere has infinite conductivity. To be zero at the ionosphere, which is equivalent to assuming \( E = 0 \) everywhere, we take the boundary condition on the field displacement as polarized east-west and radially at the equator, respectively. As a boundary condition we take the field displacement signals polarized east-west and radially at the equator, respectively.

In this section, the effect of magnetic field geometry on eigenperiods of standing Alfvén waves is illustrated for the fundamental mode toroidal oscillation with a density \( n = 1 \) proton/cm\(^3\) everywhere along the field line. Periods for other mass densities can be determined by multiplying the mass density of the plasma in amu/cm\(^3\) everywhere along the field line.

Assuming the nondipolar magnetic field model is symmetric about the noon-midnight meridian, only local times from midnight to noon are used. There are substantial deviations between the model and dipole periods at high latitudes, and periods can be larger or smaller than the dipole value, depending on latitude and local time. At any particular latitude the model period increases from noon to midnight. The steepest increase in period with latitude occurs at midnight.

As a result, the field lines are symmetric about the magnetic equatorial plane. We have tested the field line tracing procedures in our calculation by comparing our results with plots given in Olson and Pfitzer (unpublished manuscript, 1977) of \( \Delta B(B\ model-B\ dipole)\) in the noon-midnight meridian, \( |B|\) in the equatorial plane, the equatorial intercept of field lines from various magnetic latitudes on the earth's surface, and the earth surface field line intercepts from synchronous orbit as a function of local time. In all cases, agreement with the calculations given by Olson and Pfitzer was excellent.

We formulated our solution to (9), using either a dipole field or the Olson-Pfitzer model, to allow for variation of several input parameters. These parameters include the position of the field line, which can be specified by a single point, either on the ground or in space; the mass density at the equatorial crossing point of the field line; and the density index. In addition, one may choose to solve for any harmonic for polarizations in the equatorial plane either parallel or perpendicular to the radial direction, which we continue to refer to as poloidal and toroidal, respectively.

**RESULTS**

In this section the effect of magnetic field geometry on eigenperiods of standing Alfvén waves is illustrated for the fundamental mode toroidal oscillation with a density \( n = 1 \) proton/cm\(^3\) everywhere along the field line. Periods for other mass densities can be determined by multiplying the result by \( \sqrt{n} \), where \( n \) is the mass density of the plasma in amu/cm\(^3\). The terms toroidal and poloidal in this model indicate signals polarized east-west and radially at the equator, respectively. As a boundary condition we take the field displacement to be zero at the ionosphere, which is equivalent to assuming the ionosphere has infinite conductivity.

Figure 1 shows the period expected as a function of magnetic latitude for several local times at the ground position of the field line. The dipole model periods, for which there is no local time variation, are shown for purposes of comparison. Since the chosen nondipolar magnetic field model is symmetric about the noon-midnight meridian, only local times from midnight to noon are used. There are substantial deviations between the model and dipole periods at high latitudes, and periods can be larger or smaller than the dipole value, depending on latitude and local time. At any particular latitude the model period increases from noon to midnight. The steepest increase in period with latitude occurs at midnight.

Figure 2 illustrates these same results in a format that emphasizes the model period deviations from the dipole period. The ordinate of this figure is the percent increase (or decrease) of the model period from the dipole period. For example, at 67°, the magnetic latitude that corresponds to \( L = 6.6 \) and maps out to synchronous orbit in a dipole, the model fundamental period is \(-130\%\) larger than the dipole period at midnight. On the other hand, at noon, at high latitudes, the model period can be more than 50\% less than the dipole period.

The previous two figures demonstrated local time effects on period expected for selected observation positions on the ground. In Figure 3 we examine the situation in space. The percentage deviation of the model periods from dipole values is shown for different local times and different radial distances in the equatorial plane. The table provides the resonant periods calculated for different \( L \) values by using a dipole field. The results here contrast strongly with those given for observing locations on the ground. Inside of \(-9 R_E\), deviations from the dipole periods are less than 10\%, and more importantly, there is little local time variation of period. Whereas on the ground at 67° magnetic latitude the diurnal variation in period caused by field geometry is larger than a factor of 2; at 6.6 \( R_E\), where the dipole field line from 67° crosses the magnetospheric equator, the computations that use the Olson-Pfitzer model show little period dependence on local time.

Another effect of a nondipolar magnetic field model is that field lines do not necessarily remain in meridian planes. The local time at which field lines leave the earth's surface can be different from the local time where the field lines intersect the

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**Fig. 1.** Fundamental toroidal mode period for geomagnetic latitudes from 60° to 74°, using the model of W. P. Olson and K. A. Pfitzer (unpublished manuscript, 1977). Periods are given for several local times from midnight to noon and compared to dipole model periods. The results are symmetric about the noon-midnight meridian, and the mass density is 1 amu/cm\(^3\) along the entire field line.

**Fig. 2.** Same as Figure 1 except the percent deviation from the dipole model results is shown.

equatorial plane, especially at high latitudes near dawn and
dusk, where the field lines are swept tailward. Figure 4 shows
the location in the equatorial plane of field lines that leave the
earth at 1.5-hour steps from midnight to noon. From each of
those local times the intercepts are given in 1° steps, starting
with 64° geomagnetic latitude. The highest latitude positions
vary as a function of local time and were selected to remain
inside the region of validity of the Olson-Pfitzer model. The
equatorial crossing point of a field line leaving the earth at a
given latitude extends to larger radial distances as one pro-
ceeds from noon to midnight. The equatorial crossing point of
a field line leaving the earth at a given local time intersects the
equatorial plane at progressively earlier local times as the lati-
tude increases, except for midnight and noon local time. The
variations of field line intercept coordinates from those ex-
pected in a dipole field are important for the interpretation of
space-ground conjugate studies.

DISCUSSION

The development of the model in this paper was partially
prompted by the need to account for the effect on resonant pe-
riods of the difference in field line length between a dipole
and a more realistic magnetic field model [Singer and Kivel-
son, 1979]. Nevertheless, it is clear from the approximate solu-
tion for the period of standing Alfvén wave resonances given
by the WKB method (\(T \sim \int ds/v_A\)) that period depends on
both the length of the field line and the Alfvén velocity along the
field line. For any reasonable density model the greatest
contribution to the integral for the period comes from the re-
igion of smallest magnetic field strength along the field line, and
differences in field line length from a dipole are often not
as important as the magnitude of the field near the geomag-
netic equator. For this reason we can qualitatively explain our
results, as seen for example in Figure 2, by examining the field
model in the equatorial region.

Figure 5 shows the \(\Delta B\) contours in the noon-midnight meri-
dian, using the Olson-Pfitzer magnetospheric model. (The fig-
ure is from Olson and Pfitzer [unpublished manuscript, 1977]
and is also shown in Walker [1979]). The high-latitude field
lines at noon that cross the magnetospheric equator near the
magnetopause have equatorial field strength larger than the
dipole value because of magnetopause currents; therefore, as
is clear from the WKB expression, the resonant period should
be less than the dipole period. This effect was observed in Fig-
ure 2 as the decrease in the period from the dipole period at
high latitudes at noon local time. Figure 5 shows that closer to
the earth at noon there is a depression of the equatorial field
strength as a result of the ring current. Consequently, the pe-
riod should be larger than the dipole period, and this effect is
also seen in Figure 2. At midnight the depression in the field
strength near the equatorial plane because of the ring and tail
currents increases the period from the dipole value. The ef-
fects at other local times can be qualitatively accounted for in
a similar manner.

The insignificant variation of period with local time from
constant radial distances in space less than \(\sim 9 R_E\) can be ex-
plained by examining Figure 6 (also from Olson and Pfitzer),
which shows contours of constant field magnitude \(|B|\) in the
equatorial plane. The local time variation in the field magni-
tude (the field strength at the equator becomes larger as one
moves at a constant radial distance from midnight to noon) is
only beginning to become significant at the \(\sim 100-\gamma\) contour
shown in the figure. However, the field line length is also be-
coming larger as one moves at a constant radial distance from
midnight to noon, and the two effects tend to counteract one
another in determining the resonant period.

To determine the effect of the magnetic field model on the
periods of standing Alfvén waves, a constant plasma mass
density of 1 amu/cm³ has been used everywhere in the mag-
netosphere. There are, of course, spatial, temporal, and com-
positional variations in the magnetospheric plasma that could
substantially alter the local time and radial or latitudinal pat-
terns that were shown to develop as a result of the realistic
magnetic field model. In fact, using typical ion density obser-
vations from the OGO 5 satellite, for different magnetospheric
regions, Warner and Orr [1979] have considered the effect of
density on local time and latitudinal variations of standing
wave periods. Their results point up the importance of consid-
ering density as well as field line configuration for determin-
ing standing wave periods. In our calculations, periods scale
as the square root of equatorial mass density, and our plots
can easily be recalibrated for different densities. We did not
allow for density variation along the field.

Figures 1, 2, and 3 compare toroidal mode (azimuthally po-
larized) wave periods that are derived by using the Olson-Pfif-

-- Fig. 3. Percentage deviation of period calculated by using the Ol-
son-Pfitzer model from that which would be determined by using the
dipole model for observations from different equatorial radial dis-
tances in the magnetosphere. Density and mode of oscillation are the
same as for Figure 1. The insert gives the dipole model period for field
lines that cross the equator at several equatorial radial distances or \(L\)
values.

Fig. 4. Location in the equatorial plane of field lines leaving the
earth at 1.5-hour steps from midnight to noon from the Olson-Pfitzer
model. From each local time the intercepts are given in 1° steps, start-
ing with 64° geomagnetic latitude.
netic latitudes, the earth's field is often substantially distorted from the dipole geometry. Walker et al. [1979], using the netic conditions; however, as we have shown at higher mag-
al. [1977] assumed a dipole field geometry. Such an assump-
tion can be well established, but assumptions have to be made 
pared to ¾LF density measurements [Webb et al., 1977]. For 
about the field geometry and density distribution along the 
alternatives, densities have been determined by ULF techniques 
ations have been used to determine electron number densities 
in the magnetosphere [Helliwell, 1965; Carpenter and Smith, 
1964]. More recently, using the notion of standing wave oscil-
lations is good at 60 ø magnetic latitude during quiet geomag-
lar field parameter contour at the equator. For example, an al-
ments are identical at all radial distances at noon and mid-
consequently, although consistent in our selection of polarization 
with respect to an Earth-based coordinate (azimuthal), the di-
rection of polarization chosen does not align with any particu-
field parameter contour at the equator. For example, an alter-
ative would be to compare periods at different radial 
distances for waves polarized in the direction of constant field 
magnitude. We are at present modifying our calculations to 
allow for the examination of perturbations in any direction.

The determination of magnetospheric plasma density from 
ground-based observations would be extremely useful as com-
parable coverage, using in situ measurements by spacecraft, is 
unlikely. In the past, VLF measurements from ground sta-
tions have been used to determine electron number densities 
in the magnetosphere [Helliwell, 1965; Carpenter and Smith, 
1964]. More recently, using the notion of standing wave oscil-
lations, densities have been determined by ULF techniques 
[e.g., Lanzerotti et al., 1975; Cummings et al., 1978] and 
compared to VLF density measurements [Webb et al., 1977]. For 
ULF pulsations, usually the period and location of the pulsa-
tion can be well established, but assumptions have to be made 
about the field geometry and density distribution along the 
field line in order to infer equatorial plasma densities. Webb et al. [1977] assumed a dipole field geometry. Such an assumption 
is good at ≤60° magnetic latitude during quiet geomag-
agnetic conditions; however, as we have shown at higher mag-
netic latitudes, the earth's field is often substantially distorted 
from the dipole geometry. Walker et al. [1979], using the 
STARE (Scandinavian Twin Auroral Radar Experiment) ra-
dar system at high magnetic latitudes (~70°), have estimated 
magnetospheric plasma density in the vicinity of pulsations by 
assuming fundamental, toroidal mode oscillations in a dipole 
field. They suggest the need for a more realistic field model 
for more accurate determination of the densities. The model 
developed in this paper would be useful for this purpose, as 
can be seen in Figure 7. Figure 7 shows the percentage devia-
tion of inferred plasma mass density from what would be de-
termined through the use of a dipole model as a function of magnetic latitude for several local times. The toroidal funda-
mental mode has been used here. Predictions from the more 
realistic magnetic field geometry show that there could be 
substantial error in the mass density determination at high lat-
itudes if a dipole model is used.

To the best of our knowledge, the only other model of 
standing Alfven waves that uses a realistic magnetic field 
model for the earth is that of Warner and Orr [1979]. Using 
the WKB approximation and the Mead and Fairfield [1975] 
magnetic field model, they clearly demonstrate the impor-
tance of considering field model and density variations for 
calculating eigenperiods on field lines. As was discussed ear-
lier, the WKB approximation is particularly poor for the funda-
mental period, which is the most probable mode, as re-
ported by many observers [Lanzerotti et al., 1972; Cummings 
et al., 1975; Singer and Kivelson, 1979]. We preferred to use 
the Olson-Pfitzer model since it has been shown to fit the 
magnetic field strength at the equator in the noon-midnight meridian much better than the Mead-Fairfield model 
[Walker, 1976; W. P. Olson and A. Pfitzer, unpublished manu-
script, 1977]. We have shown the effect of the field model on 
the local time variation of periods from ground observations 
as did Warner and Orr. We are unable to make direct com-
parisons with Warner and Orr's results because, in the pub-
lished results, tilt, Kp, and density are varied simultaneously. 
We have isolated the effects of the magnetic field model and 
also have demonstrated the local time variation in period 
from a vantage point in space. Unlike the WKB approxima-
tion, the solution to the wave equation developed in this paper 
permits accurate determination of not only the period but also 
the perturbation fields along a field line.

A worthwhile direction for the further investigation of 
standing wave models is to combine the model which uses re-

Fig. 5. $\Delta B$ contours ($B_{\text{Olson-Pfitzer}} - B_{\text{dipole}}$) in the noon-mid-night meridian plane (Tilt=0°).

Fig. 6. Calculated from the model of Olson and Pfitzer (as de-
scribed in W. P. Olsen and A. Pfitzer, unpublished manuscript, 1977) 
Contours of constant field magnitude $|B|$ in the equatorial plane. $B$ is 
given in gamma.
alistic field geometry in this paper with different density models in space and different ionospheric boundary conditions [Newton et al., 1978]. The capability to adjust the ionospheric boundary conditions at the two ends of the field line to account for differing degrees of illumination should be included [Allan and Knox, 1979a, b]. The effect of a tilted magnetic field, such as considered by Warner and Orr [1979], should allow determination of the latitude of nodes of resonant oscillations and the amplitude structure along a field line for different harmonics. These results would be important for understanding seasonal variations of pulsation observations.

**Conclusions**

The linearized transverse wave equation for low-frequency propagation in a cold, collisionless, magnetized plasma has been solved in an arbitrary magnetic field geometry. We obtained solutions for the symmetric toroidal and highly asymmetric poloidal mode standing wave oscillations in both dipole and Olson-Pfitzer field models. The realistic description of the earth's field given by the Olson-Pfitzer model leads to substantial differences in eigenfrequencies from those determined by using dipole models. Therefore, our model should improve our ability to analyze pulsation observations.

The use of the model described in this paper is most important for studying oscillations on field lines which extend to large radial distances where large deviations from a dipole configuration occur. It has been shown that from a fixed latitude on the ground, the period of oscillation varies with local time and can be larger or smaller than the period calculated by using a dipole model. Local time variations of period from a fixed radial distance in space differ from observations from a fixed magnetic latitude. Due to field geometry, the local time variations of period from a fixed radial distance at the magnetospheric equator are not significant until large radial distances are reached. In particular, there are insignificant variations of period with local time at synchronous orbit.

The model developed is important for ground-satellite conjugate studies, since field lines going through a particular position in space can intersect the ground at a different local time and latitude than would be found by using a dipole. In addition, it has been shown that the diagnostic technique that uses ULF waves to determine mass density in space from ground observations requires a realistic field model, since the use of a dipole model can introduce errors, especially at high latitudes.

Finally, since the equation for standing Alfven waves derived in this paper can be used easily with any magnetic field model, we could replace the earth's field with that of Jupiter or Saturn. Prediction of pulsation periods in planetary magnetospheres will be one technique for determining if observed magnetic perturbations are standing Alfven waves.

**Appendix: Validity of the Linearized Momentum Equation (4)**

Equation (4) takes the form

\[ \rho (\frac{\partial^2 \mathbf{v}}{\partial t^2}) = \mathbf{j} \times \mathbf{B}_0 \]

It explicitly ignores a force \( \mathbf{j} \times \mathbf{b} \), which must be present in a plasma carrying a zero-order current across the background field. The Olson-Pfitzer field model contains such currents (the ring current), and the force must be present. On order of magnitude grounds, however, it is unlikely to be significant, as the current density in the Olson-Pfitzer model does not exceed \( 1.5 \times 10^{-7} \) A/m². Using a crude estimate, \( j \sim b/\eta_0 \), where \( \eta_0 = \) scale length parallel to \( \mathbf{B} \), one finds \( j \times \mathbf{b} \) should exceed the neglected term by over an order of magnitude at synchronous orbit. Inside this orbit the neglect seems fully justified. Out to 12 Re, the neglected term is everywhere less than \( \pm 25\% \) of the term we have used.

Some further comments can be made. The force \( \mathbf{j} \times \mathbf{b} \) is parallel to \( \mathbf{B}_0 \) if the wave perturbation is transverse and thus its precise effect depends on the conditions governing ion and electron dynamics along the field. Unless conditions were just such that strong coupling could be established with an acoustic type of mode standing along the field line, the force only produces small net displacements of plasma back and forth along the field at the wave frequency. Equation (4) also ignores an implicit feature of the Olson-Pfitzer field. There must be hot plasma gradients present to provide the zero-order momentum balance;

\[ \nabla P = \mathbf{j} \times \mathbf{B}_0 \]

if the pressure is isotropic. The plasma must be displaced by the wave, and even if the wave motion does not compress the plasma, changes in pressure will be produced by the convection of gradients by the wave. Southwood [1977] pointed out that this could lead to compressional magnetic field changes in the wave, as is often seen. We have made no effort to examine this further, but it will need consideration if this line of approach is continued.

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